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Keller–Osserman a priori estimates and the Harnack inequality for quasilinear elliptic and parabolic equations with absorption term

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ABSTRACT

In this article we study quasilinear equations model of which are

$$-\sum_{i=1}^{n}\left(|u_{x_i}|^{p_i-2}u_{x_i}\right)_{x_i}+f(u)=0, \quad u\geq 0,$$

$$\frac{\partial u}{\partial t}-\sum_{i=1}^{n}\left(u^{(m_i-1)(p_i-1)}|u_{x_i}|^{p_i-2}u_{x_i}\right)_{x_i}+f(u)=0, \quad u\geq 0.$$

Despite of the lack of comparison principle, we prove a priori estimates of Keller–Osserman type. Particularly under some natural assumptions on the function f, for nonnegative solutions of p-Laplace equation with absorption term we prove an estimate of the form

$$\int_0^{u(x_0)} f(s)ds \leq cr^{-p}u^p(x_0), \quad x_0\in \Omega, \qquad B_{8r}(x_0)\subset \Omega,$$

with constant c independent of u, using this estimate we give a simple proof of the Harnack inequality. We prove a similar result for the evolution p-Laplace equation with absorption.

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1. Introduction and main results

The qualitative behavior of large solutions of elliptic and parabolic equations has been investigated by many authors starting from the seminal papers of Keller [11] and Osserman [27]. For example, for the elliptic equation with an absorption term

$$-\triangle_p u + f(u) = 0,$$

any nonnegative solution u satisfies

$$u(x) \le c\, dist(x, \partial\Omega)^{-\frac{p}{q-p+1}}, \qquad f(u) = u^q, \quad q > p-1$$
$$u(x) \le c\, |\ln\, dist(x, \partial\Omega)|, \qquad f(u) = e^u.$$

Estimates of this type play a crucial role in the theory of existence or nonexistence of large solutions, in the problems of removable singularities for solutions to elliptic and parabolic equations. Up to our knowledge all the known estimates for large solutions to elliptic and parabolic equations are related with equations for which some comparison properties hold. We refer to [18,22,31,37] for an account of these results and references therein.

Anisotropic elliptic and parabolic equations have been the object of very few works because in general such properties do not hold. The main ones concern equations only in the precise choice of absorption term $f(u) = u^q$ see [1,5,25,26,33–35,40,38,39].

In this article we give a proof of the Keller–Osserman a priori estimates for solutions to anisotropic elliptic and parabolic equations with absorption. Using these estimates we give a simple proof of the Harnack inequality for solutions of p-Laplace and evolution p-Laplace equations with absorption term.

The first main result of this paper is an a priori estimate of the Keller–Osserman type for nonnegative solutions to elliptic equations

$$-\, div\, A(x, \nabla u) + a_0(u) = 0, \quad x \in \Omega, \tag{1.1}$$

where Ω is a bounded domain in R^n, $n \ge 2$.

We suppose that the functions $A = (a_1, \dots, a_n)$ and a_0 satisfy the Caratheodory conditions and the following structure conditions hold

$$\begin{aligned} & A(x,\xi)\xi \ge \nu_1 \sum_{i=1}^n |\xi_i|^{p_i}, \\ & |a_i(x,\xi)| \le \nu_2 \left(\sum_{j=1}^n |\xi_j|^{p_j} \right)^{1-\frac{1}{p_i}}, \quad i = \overline{1,n} \\ & a_0(u) \ge \nu_1 f(u), \end{aligned} \tag{1.2}$$

where ν_1, ν_2 are positive constants and f is continuous positive function.

We say that u is a weak solution to Eq. (1.1) in Ω if $u \in W^{1,p_1,p_2,\dots,p_n}(\Omega) = \big\{ u \in W^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i},\ i = \overline{1,n} \big\}$ and the integral identity

$$\int_\Omega \{ A(x, \nabla u)\nabla\varphi + a_0(u)\varphi \}\, dx = 0 \tag{1.3}$$

holds for any $\varphi \in \overset{o}{W}{}^{1,p_1,p_2,\dots,p_n}(\Omega)$.

Let $x^{(0)} \in \Omega$, for any $\theta_1, \theta_2, \dots, \theta_n > 0$, $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ we define $Q_\theta(x^{(0)}) := \{x : |x_i - x_i^{(0)}| < \theta_i,\ i = \overline{1,n}\}$ and set $F(u) = \int_0^u f(s)ds$, $\delta(u) = \frac{F(u)}{f(u)}$, $M(\theta) = \sup_{Q_\theta(x^{(0)})} u$, $\delta(\theta) = \sup_{Q_\theta(x^{(0)})} \delta(u)$, $F(\theta) = \sup_{Q_\theta(x^{(0)})} F(u)$, $N(\theta) = \max\{M(\theta), \delta(\theta)\}$.

Theorem 1.1. *Let the conditions (1.2) be fulfilled and u be a nonnegative weak solution to Eq. (1.1) in Ω, assume also that*

$$1 < p_1 \le p_2 \le \cdots \le p_n \le \frac{np}{n-p}, \quad \frac{1}{p} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i},\ p < n. \tag{1.4}$$

Let $x^{(0)} \in \Omega$, fix $\sigma \in (0,1)$, $\theta_i \in (0,\theta_n)$ for $i \in I' = \{i = \overline{1,n} : p_i < p_n\}$ and $\theta_i = \theta_n$ for $i \in I'' = \{i = \overline{1,n} : p_i = p_n\}$, then there exist positive numbers c_1, c_2 depending only on $n, \nu_1, \nu_2, p_1, \dots, p_n$ such that either

$$u(x^{(0)}) \le \sum_{i\in I'} \left(\theta_i^{-1}\theta_n^{\frac{p_n}{p_i}}\right)^{\frac{p_i}{p_n-p_i}} \tag{1.5}$$

or

$$F(\sigma\theta) \le c_1(1-\sigma)^{-c_2}\theta_n^{-p_n}\left\{\delta(\theta)N^{p_n-1}(\theta) + \delta^p(\theta) + N^{p_n}(\theta)\sum_{i=1}^{n}\left(\frac{\delta(\theta)}{N(\theta)}\right)^{\frac{(p_i-1)p}{p-1}}\right\}, \tag{1.6}$$

for all $Q_{8\theta}(x^{(0)}) \subset \Omega$.

On the other hand, if I' is empty, i.e. $p = p_1 = p_2 = \cdots = p_n$, then there exist positive numbers c_3, c_4 depending only on p, n, ν_1, ν_2 such that

$$F(\sigma\theta) \le c_3(1-\sigma)^{-c_4}\theta^{-p}\delta(\theta)\left(M^{p-1}(\theta) + \delta^{p-1}(\theta)\right), \tag{1.7}$$

for all $\sigma \in (0,1)$ and $\theta > 0$, such that $B_{8\theta}(x^{(0)}) \subset \Omega$, where $F(\theta) = \sup_{B_\theta(x^{(0)})} F(u)$, $\delta(\theta) = \sup_{B_\theta(x^{(0)})} \delta(u)$, $M(\theta) = \sup_{B_\theta(x^{(0)})} u$ and $B_\theta(x^{(0)}) = \{x : |x_i - x_i^{(0)}| < \theta,\ i = \overline{1,n}\}$.

Remark 1.1. Condition (1.4) implies the local boundedness of solutions [12]. The condition is sharp as there are unbounded solutions to (1.1) if its condition is violated (cf. [7,20]).

Remark 1.2. Using the comparison theorem and radial type solutions, under the additional condition that $f(u)$ is nondecreasing, inequality of the type (1.7) was proved in [15,16]. Inequality (1.7) implies the Keller–Osserman condition for large solution. This result has been proved in [17].

It is of interest to have more precise sub-estimate of solutions. For this we use the following additional condition. We say that nondecreasing continuous function ψ satisfies the condition (A) if for any $\varepsilon \in (0,1)$ there exists $u_0(\varepsilon) \ge 1$ such that

$$\psi(\varepsilon u) \le \varepsilon^{\mu}\psi(u), \tag{A}$$

with some $\mu > 0$ and for all $u \ge u_0(\varepsilon)$.

Proposition 1.1. *Let the conditions (1.2), (1.4) be fulfilled and u be a nonnegative weak solution to Eq. (1.1). Assume also that $\lim_{x\to\partial\Omega} u(x) = +\infty$, and with some $0 \le a \le 1$ and $c > 0$ there holds*

$$\delta(u) \le cu^a. \tag{1.8}$$

Set $b = \max\left(a + p_n - 1,\ p_n + (a-1)(p_1-1)\frac{p}{p-1}\right)$ and let $\psi(u) = u^{-1}F^{\frac{1}{b}}(u)$ satisfy the condition (A). Let $x^{(0)} \in \Omega$ and $8\rho = dist(x^{(0)}, \partial\Omega)$, fix $\theta_i \in (0,\rho)$ for $i \in I'$, then there exists a positive number c_5 depending only on $n, \nu_1, \nu_2, p_1, \dots, p_n$ and c such that either

$$u(x^{(0)}) \le \sum_{i\in I'} \left(\theta_i^{-1}\rho^{\frac{p_n}{p_i}}\right)^{\frac{p_i}{p_n-p_i}}, \tag{1.9}$$

or

$$F(u(x^{(0)})) \leq c_5\, dist^{-p_n}(x^{(0)}, \partial\Omega) u^b(x^{(0)}). \tag{1.10}$$

On the other hand, if I' is empty, i.e. $p = p_1 = p_2 = \cdots = p_n$ and $\psi(u) = u^{-1} F^{\frac{1}{p+a-1}}(u)$ satisfies the condition (A), then

$$F(u(x^{(0)})) \leq c_6\, dist^{-p}(x^{(0)}, \partial\Omega) u^{p+a-1}(x^{(0)}) \tag{1.11}$$

with positive constant c_6 depending only on p, n, ν_1, ν_2 and c.

A function which satisfies the conditions of Proposition 1.1 with $a = 1$ is $f(u) = u^q, q > p_n - 1$. Assuming for simplicity that $dist(x^{(0)}, \partial\Omega) = |x^{(0)}|$ and choosing θ_i from the conditions $\left(\theta_i^{-1} \rho^{\frac{p_n}{p_i}}\right)^{\frac{p_i}{p_n - p_i}} = \rho^{-\frac{p_n}{q-p_n+1}}$, i.e. $\theta_i = \rho^{\frac{p_n}{p_i}\frac{q-p_i+1}{q-p_n+1}}, i \in I'$, from (1.9), (1.10) we obtain

$$u(x^{(0)}) \leq c \left(\sum_{i=1}^{n} |x_i^{(0)}|^{\frac{p_i}{q-p_i+1}} \right)^{-1},$$

which was proved in [33].

Another example of the function f, which satisfies the conditions of Proposition 1.1 with $a = 0$, is $f(u) = e^u$. Assuming that $8\rho = dist(x^{(0)}, \partial\Omega)$ and choosing θ_i from the conditions $\left(\theta_i^{-1} \rho^{\frac{p_n}{p_i}}\right)^{\frac{p_i}{p_n - p_i}} = \ln(1 + \rho^{-p_n})$, $i \in I'$, from (1.9), (1.10) we obtain

$$u(x^{(0)}) \leq c |\ln\, dist(x^{(0)}, \partial\Omega)|,$$

in the anisotropic case it seems that this estimate is new.

To prove the Harnack inequality we also need the following additional condition. We say that continuous function Ψ satisfies the condition (B) if there exists $\mu > 0$ such that

$$\Psi(\varepsilon u) \leq \varepsilon^{\mu}\, \Psi(u), \tag{B}$$

for all $\varepsilon \in (0, 1)$, and for all $u > 0$.

Proposition 1.2. *Let the condition (1.2) be fulfilled, $1 < p = p_1 = p_2 = \cdots = p_n < n$ and u be a nonnegative weak solution to (1.1), and let $\Psi(u) = u^{-1} F^{\frac{1}{p}}(u)$ satisfy the condition (B). Let $x^{(0)} \in \Omega$ and $B_{8\rho}(x^{(0)}) \subset \Omega$, then there exists a positive number c_7 depending only on n, ν_1, ν_2, p such that*

$$F(u(x)) \leq c_7 \rho^{-p} u^p(x), \tag{1.12}$$

for almost all $x \in B_\rho(x^{(0)})$.

Remark 1.3. If $f(u) = u^q f_1(u)$, where f_1 is nondecreasing, continuous function and $q > p - 1$, then the function $\Psi(u) = u^{-1} F^{\frac{1}{p}}(u)$ satisfies the condition (B) with $\mu = \frac{q-p+1}{p} > 0$.

Remark 1.4. If $f(u) = u^{p-1} f_1(u)$, where f_1 satisfies the condition (B) with some $\mu_1 > 0$, then the function $\Psi(u) = u^{-1} F^{\frac{1}{p}}(u)$ satisfies the condition (B) with $\mu = \frac{\mu_1}{p} > 0$. A simple example of the function f_1, which satisfies the condition (B) with $\mu_1 = 1$ is a function $f_1(u) = \int_0^u \widetilde{f}_1(s)\, ds$, where $\widetilde{f}_1$ is nondecreasing, continuous function.

We also note, that if f is as in Proposition 1.2, then u is uniformly bounded.

The next main result of this paper is a priori estimate of the Keller–Osserman type for solutions of the equation

$$u_t - div\, A(x,t,u,\nabla u) + a_0(u) = 0, \quad (x,t) \in \Omega_T, \tag{1.13}$$

where $\Omega_T = \Omega \times (0,T)$, $0 < T < \infty$.

We suppose that the functions $A = (a_1,\dots,a_n)$ and a_0 satisfy the Caratheodory conditions and the following structure conditions hold

$$\begin{aligned} &A(x,t,u,\xi)\xi \geq \nu_1 \sum_{i=1}^n |u|^{(m_i-1)(p_i-1)}|\xi_i|^{p_i}, \\ &|a_i(x,t,u,\xi)| \leq \nu_2 u^{(m_i-1)\frac{p_i-1}{p_i}} \left(\sum_{j=1}^n |u|^{(m_j-1)(p_j-1)}|\xi_j|^{p_j}\right)^{1-\frac{1}{p_i}}, \quad i=\overline{1,n}, \\ &a_0(u) \geq \nu_1 f(u), \end{aligned} \tag{1.14}$$

with positive constants ν_1, ν_2 and continuous, positive function $f(u)$ and

$$2 < p_1 \leq \dots \leq p_n, \ \min_{1\leq i\leq n} m_i > 1, \quad \max_{1\leq i\leq n} m_i(p_i-1) \leq 1 + \frac{\kappa}{n}, \ p < n, \tag{1.15}$$

where $\kappa = n(p(m-d)-1)+p$, $d = \frac{1}{n}\sum_{i=1}^n \frac{m_i}{p_i}$, and assume without loss of generality, that $m_n = \max_{1\leq i\leq n} m_i$.

We will write $V_{p,m}(\Omega_T)$ for the class of functions $\varphi \in C(0,T,L^2(\Omega))$ with $\sum_{i=1}^n \iint_{\Omega_T} |\varphi|^{(m_i-1)(p_i-1)}|\varphi_{x_i}|^{p_i}\, dxdt < \infty$. We say that u is a weak solution to (1.13) if we have an inclusion $u \in V_{p,m}(\Omega_T)$ and for any interval $(t_1,t_2) \subset (0,T)$ the integral identity

$$\int_\Omega u\varphi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega \{-u\varphi_t + A(x,t,u,\nabla u)\nabla\varphi + a_0(u)\varphi\}\, dx\, dt = 0 \tag{1.16}$$

holds for any $\varphi \in \overset{o}{V}_{p,m}(\Omega_T)$.

Remark 1.5. Condition (1.15) implies the local boundedness of weak solutions to Eq. (1.13) [13].

Let $(x^{(0)},t^{(0)}) \in \Omega_T$, for any $\tau, \theta_1, \theta_2, \dots, \theta_n > 0$, $\theta = (\theta_1,\dots,\theta_n)$ we define $Q_{\theta,\tau}(x^{(0)},t^{(0)}) := \{(x,t) : |t-t^{(0)}| < \tau,\ |x_i - x_i^{(0)}| < \theta_i,\ i=\overline{1,n}\}$ and set $M(\theta,\tau) := \sup_{Q_{\theta,\tau}(x^{(0)},t^{(0)})} u$, $\delta(\theta,\tau) := \sup_{Q_{\theta,\tau}(x^{(0)},t^{(0)})} \delta(u)$, $\Phi(\theta,\tau) := \sup_{Q_{\theta,\tau}(x^{(0)},t^{(0)})} \Phi(u)$, $\Phi(u) = \int_0^u g(s)ds$, $g(s) = s^{m_n-1}f(s)$.

Theorem 1.2. *Let the conditions (1.14), (1.15) be fulfilled and u be a nonnegative weak solution to Eq. (1.13), assume also that $f \in C^1(R^1_+)$ and $f'(u) \geq 0$. Let $(x^{(0)},t^{(0)}) \in \Omega_T$, fix $\sigma \in (0,1)$, $\tau \in (0, \min(\theta_n^{p_n}, t^{(0)}, T-t^{(0)}))$, $\theta_i \in (0,\theta_n)$ for $i \in I' = \{i = \overline{1,n} : m_i(p_i-1) < m_n(p_n-1)\}$ and $\theta_i = \theta_n$ for $i \in I'' = \{i=\overline{1,n} : m_i(p_i-1) = m_n(p_n-1)\}$, then there exist positive numbers c_8, c_9 depending only on $n, \nu_1, \nu_2, m_1, \dots, m_n, p_1, \dots, p_n$ such that either*

$$u(x^{(0)},t^{(0)}) \leq (\tau^{-1}\rho^{p_n})^{\frac{1}{m_n(p_n-1)-1}} + \sum_{i\in I'} \left(\theta_i^{-1}\theta_n^{\frac{p_n}{p_i}}\right)^{\frac{p_i}{m_n(p_n-1)-m_i(p_i-1)}}, \tag{1.17}$$

or

$$\Phi(\sigma\theta,\sigma\tau) \leq c_8(1-\sigma)^{-c_9}\theta_n^{-p_n}\delta(\theta,\tau)M^{m_np_n-1}(\theta,\tau). \tag{1.18}$$

On the other hand, if I' is empty, i.e. $m_1(p_1-1) = m_2(p_2-1) = \dots = m_n(p_n-1)$, then either

$$u(x^{(0)},t^{(0)}) \leq (\tau^{-1}\theta_n^{p_n})^{\frac{1}{m_n(p_n-1)-1}}, \tag{1.19}$$

or (1.19) *holds true.*

Completely similar to Proposition 1.1 we obtain

Proposition 1.3. *Let the conditions* (1.14), (1.15) *be fulfilled, u be a nonnegative weak solution to* (1.13), *$f \in C^1(R^1_+)$ and $f'(u) \geq 0$. Let $\partial\Omega_T$ be the parabolic boundary of Ω_T, assume also that $\lim_{(x,t)\to\partial\Omega_T} u(x,t) = +\infty$ and with some $0 \leq a \leq 1$ and $c > 0$ there holds*

$$\delta(u) = \frac{F(u)}{f(u)} \leq c\, u^a.$$

Let $\psi(u) = u^{-1}\Phi^{\frac{1}{m_n p_n + a - 1}}(u)$ satisfy the condition (A). *Let $(x^{(0)}, t^{(0)}) \in \Omega_T$ and $8\rho = dist(x^{(0)}, \partial\Omega)$. Fix $\tau \in (0, \min(\rho^{p_n}, t^{(0)}, T - t^{(0)}))$ and $\theta_i \in (0, \rho)$ for $i \in I'$, then there exists a positive number c_{10} depending only on $n, \nu_1, \nu_2, m_1, \ldots, m_n, p_1, \ldots, p_n$ and c, such that either* (1.18) *holds, or*

$$\Phi(u(x^{(0)}, t^{(0)})) \leq c_{10}\theta_n^{-p_n} u^{m_n p_n + a - 1}(x^{(0)}, t^{(0)}). \tag{1.20}$$

On the other hand if I' is empty, i.e. $m_1(p_1 - 1) = m_2(p_2 - 1) = \cdots = m_n(p_n - 1)$ and $\psi(u) = u^{-1}\Phi^{\frac{1}{m_n p_n + a - 1}}(u)$ satisfies the condition (A), *then either* (1.19) *holds, or* (1.20) *holds true.*

A first main example of the function f, which satisfies the conditions of Proposition 1.3 with $a = 1$ is $f(u) = u^q, q > m_n(p_n - 1)$. Assuming for simplicity that $dist(x^{(0)}, \partial\Omega) = |x^{(0)}|$, and choosing $\tau,\ \theta_i,\ i \in I'$ from the conditions $\left(\tau^{-1}\rho^{p_n}\right)^{\frac{1}{m_n(p_n-1)-1}} = \rho^{-\frac{p_n}{q - m_n(p_n-1)}}$, i.e. $\tau = \rho^{\frac{p_n(q-1)}{q - m_n(p_n-1)}}$, $\left(\theta_i^{-1}\rho^{\frac{p_n}{p_i}}\right)^{\frac{p_i}{m_n(p_n-1) - m_i(p_i-1)}} = \rho^{-\frac{p_n}{q - m_n(p_n-1)}}$, i.e. $\theta_i = \rho^{\frac{p_n}{p_i}\frac{q - m_i(p_i-1)}{q - m_n(p_n-1)}}$, from (1.17), (1.20) we obtain an estimate

$$u(x^{(0)}, t^{(0)}) \leq c\left(\sum_{i=1}^n |x_i^{(0)}|^{\frac{p_i}{q - m_i(p_i-1)}} + (t^{(0)})^{\frac{1}{q-1}}\right)^{-1}, \tag{1.21}$$

in the case $m_1 = m_2 = \cdots = m_n = 1$ it was proved in [34].

Another example of the function f, which satisfies the conditions of Proposition 1.3 with $a = 0$, is $f(u) = e^u$. Assuming that $(8\rho)^{p_n} = |x^{(0)}|^{p_n} + t^{(0)}$ and choosing $\tau, \theta_i, i \in I'$ from the conditions

$$\left(\tau^{-1}\rho^{p_n}\right)^{\frac{1}{m_n(p_n-1)-1}} = \left(\theta_i^{-1}\rho^{\frac{p_n}{p_i}}\right)^{\frac{p_i}{m_n(p_n-1) - m_i(p_i-1)}} = \ln(1 + \rho^{-p_n}),$$

from (1.17), (1.20) we obtain an estimate

$$u(x^{(0)}, t^{(0)}) \leq c|\ln(|x^{(0)}|^{p_n} + t^{(0)})|,$$

in the anisotropic case it seems that this estimate is new.

Similar to Proposition 1.2 we get

Proposition 1.4. *Let the conditions* (1.14) *be fulfilled, $m_1 = m_2 = \cdots = m_n = 1$, $2 < p = p_1 = p_2 = \cdots = p_n$ and u be a nonnegative weak solution to* (1.13), *assume also that $f \in C^1(R^1_+), f' \geq 0$ and let $\Psi(u) = u^{-1}\Phi^{\frac{1}{p}}(u)$ satisfy the condition* (B). *Let $(x^{(0)}, t^{(0)}) \in \Omega_T$ and $Q_{8\rho,8\tau}(x^{(0)}, t^{(0)}) \subset \Omega_T$, then there exists a positive constant c_{11} depending only on n, ν_1, ν_2, p such that either*

$$u(x^{(0)}, t^{(0)}) \leq (\tau^{-1}\rho^p)^{\frac{1}{p-2}}, \tag{1.22}$$

or

$$\Phi(u(x,t)) \leq c_{11}\rho^{-p}u^p(x,t), \tag{1.23}$$

for almost all $(x,t) \in Q_{\rho,\tau}(x^{(0)}, t^{(0)})$.

Now we give a simple extension of the Keller–Osserman type estimates to the proof of Harnack type inequality for nonnegative solutions of p-Laplace and evolution p-Laplace equations with absorption term.

Harnack inequality is one of the most important results in the qualitative theory of elliptic and parabolic equations. We refer to [2,3,23,24,32] for an account of these results. Generalized Harnack inequality was proved in [10,9] for the equation of the type

$$-\triangle u + f(u) = 0,$$

$f \geq 0$ nondecreasing function and it has the form

$$\int_{\inf_{B_\rho(x^{(0)})} u}^{\sup_{B_\rho(x^{(0)})} u} \frac{ds}{\rho\sqrt{F(s)} + s} \leq c, \qquad B_{8\rho}(x^{(0)}) \subset \Omega \tag{1.24}$$

with constant c independent of u.

Theorem 1.3. *Let u be a nonnegative weak solution to Eq. (1.1), let the conditions (1.2) be fulfilled and $1 < p = p_1 = p_2 = \cdots = p_n < n$, assume also that $a_0(u) \leq \nu_2 f(u)$, and let f is nondecreasing and $\Psi(u) = u^{-1}F^{\frac{1}{p}}(u)$ satisfies the condition (B). Then there exists positive number c_{12} depending only on ν_1, ν_2, n, p and independent of u such that*

$$\sup_{B_\rho(x^{(0)})} u \leq c_{12} \inf_{B_\rho(x^{(0)})} u, \tag{1.25}$$

for all $B_{8\rho}(x^{(0)}) \subset \Omega$.

Remark 1.6. Harnack inequality (1.25) implies the strong maximum principle, see [4,29,30,28,36].

Note also that Theorem 1.3 implies the existence of a radius $\rho > 0$ such that $\sup_{B_\rho(x^{(0)})}$ is uniformly bounded by $u(x^{(0)})$ and a constant which depends only on n, p, ν_1, ν_2.

Remark 1.7. If $p = 2$ by Proposition 1.2, we have

$$F(\sup_{B_\rho(x^{(0)})} u) \leq \gamma\rho^{-2}(\sup_{B_\rho(x^{(0)})} u)^2,$$

which implies for every $0 < s < \sup_{B_\rho(x^{(0)})} u$

$$\Psi(s) = \frac{\sqrt{F(s)}}{s} \leq \Psi(\sup_{B_\rho(x^{(0)})} u) \leq \gamma\rho^{-1},$$

and hence (1.24) yields (1.25).

Theorem 1.4. *Let u be a nonnegative weak solution to Eq. (1.13) in Ω_T, let the conditions (1.14) be fulfilled and $2 < p_1 = p_2 = \cdots = p_n$, $m_1 = m_2 = \cdots = m_n = 1$, assume also that, $a_0(u) \leq \nu_2 f(u)$, and let $f \in C^1(R^1_+)$, $f \geq 0$ and $\Psi(u) = u^{-1}F^{\frac{1}{p}}(u)$ satisfy the condition (B). Then there exist positive constants c_{13}, c_{14} depending only on ν_1, ν_2, n, p and independent of u such that*

$$u(x^{(0)}, t^{(0)}) \leq c_{13} \inf_{B_\rho(x_0)} u(x, t^{(0)} + \tau), \quad \tau = \rho^p \left(\frac{c_{14}}{u(x^{(0)}, t^{(0)})}\right)^{p-2}, \tag{1.26}$$

for all $Q_{8\theta,8\tau}(x^{(0)}, t^{(0)}) \subset \Omega_T$.

Formulations of Theorems 1.3 and 1.4 are the same as in [2,3,23,24,32], however, due to the presence of lower order term the results of [2,3,23,24,32] cannot be used. If $f(u) = u^q, q > p - 1$, then the Harnack inequality is a simple consequence of the Keller–Osserman estimate (see, for example [14,21]). The main novelty of our results is that the constants c_{12}, c_{13}, c_{14} are independent of u. The method that we are using is a De Giorgi method. In the parabolic case we also use the well-known intrinsic scaling technique originally introduced by Di Benedetto [2].

The rest of the paper contains the proof of the above theorems.

2. Keller–Osserman a priori sub-estimates for elliptic equations. Proof of Theorem 1.1 and Propositions 1.1 and 1.2

2.1. Auxiliary propositions

The following lemmas will be used in the sequel. The first one is the well-known embedding lemma (see [19, chap. 2]).

Lemma 2.1. *Let $\Omega \in R^n$, $n \geq 2$ be a bounded domain, $u \in \overset{o}{W}{}^{1,1}(\Omega)$, then the following inequality holds*

$$\|u\|_{L^q(\Omega)} \leq \gamma \prod_{i=1}^{n} \left(\int_{\Omega} |u_{x_i}| dx \right)^{\frac{1}{n}}, \quad q = \frac{n}{n-1},$$

where the positive constant γ depends only on n.

In what follows we will frequently use the following lemma [19, chap. 2].

Lemma 2.2. *Let $\{y_j\}_{j \in N}$ be a sequence of nonnegative numbers such that for any $j = 0, 1, 2, \ldots$ the inequality*

$$y_{j+1} \leq C b^j y_j^{1+\varepsilon}$$

holds with positive $\varepsilon, C > 0, b > 1$. Then the following estimate is true

$$y_j \leq C^{\frac{(1+\varepsilon)^j - 1}{\varepsilon}} b^{\frac{(1+\varepsilon)^j - 1}{\varepsilon^2} - \frac{j}{\varepsilon}} y_0^{(1+\varepsilon)^j}.$$

Particularly, if $y_0 \leq C^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}}$, then $\lim_{j \to \infty} y_j = 0$.

2.2. Local energy estimates

The proof of sub-bounds stated in the previous section is based on local energy estimates. We provide the proof of (1.6), assuming without loss that

$$1 < p_1 \leq p_2 \leq \cdots \leq p_{n-1} < p_n \leq \frac{np}{n-p}, \tag{2.1}$$

while the proof of (1.7) is completely similar. In what follows γ stands for a constant depending only on $n, \nu_1, \nu_2, p_1, \ldots, p_n$ which may vary from line to line. If $Q_\eta(\overline{x}) \subset \Omega$ we let ζ denote a nonnegative piecewise smooth cutoff function vanishing on the boundary of $Q_\eta(\overline{x})$.

Lemma 2.3. *Let u be a nonnegative weak solution to Eq. (1.1) and let conditions (1.2), (2.1) hold. Then for every $Q_\eta(\overline{x}) \subset \Omega$ and for every $k > 0$*

$$\sum_{i=1}^{n} \int_{A_{k,\eta}} f(u) |u_{x_i}|^{p_i} \zeta^{p_n} dx + \int_{A_{k,\eta}} (F(u) - k)_+ f(u) \zeta^{p_n} dx$$
$$\leq \gamma \sum_{i=1}^{n} \int_{A_{k,\eta}} (F(u) - k)_+ \delta^{p_i - 1}(u) |\zeta_{x_i}|^{p_i} dx, \tag{2.2}$$

where $A_{k,\eta} = \{x \in Q_\eta(\overline{x}) : F(u) > k\}$.

Proof. Testing identity (1.3) by $\varphi = (F(u)-k)_+\zeta^{p_n}$, using conditions (1.2) we obtain

$$\sum_{i=1}^{n}\int_{A_{k,\eta}} f(u)|u_{x_i}|^{p_i}\zeta^{p_n}dx + \int_{A_{k,\eta}}(F(u)-k)_+ f(u)\zeta^{p_n}dx$$
$$\le \gamma\sum_{i=1}^{n}\int_{A_{k,\eta}}\left(\sum_{j=1}^{n}|u_{x_j}|^{p_j}f(u)\zeta^{p_n}\right)^{1-\frac{1}{p_i}}(F(u)-k)_+ f^{\frac{1}{p_i}-1}(u)|\zeta_{x_i}|\zeta^{\frac{p_n}{p_i}-1}dx.$$

From this, using the Young inequality we arrive at the required (2.2). □

2.3. *Proof of Theorem 1.1*

Consider a cylinder $Q_\theta(x^{(0)})$ and let $\bar{x}$ be an arbitrary point in $Q_{\sigma\theta}(x^{(0)})$. If $u(x^{(0)}) \ge \sum_{i=1}^{n-1}\left(\theta_i^{-1}\rho^{\frac{p_n}{p_i}}\right)^{\frac{p_i}{p_n-p_i}}$, then $N(\theta) \ge \left(\theta_i^{-1}\rho^{\frac{p_n}{p_i}}\right)^{\frac{p_i}{p_n-p_i}}$ for $i=\overline{1,n-1}$, and hence $Q_\eta(\bar{x}) \subset Q_\theta(x^{(0)})$, where $\eta_i = (1-\sigma)\theta_n^{\frac{p_n}{p_i}}N^{-\frac{p_n-p_i}{p_i}}(\theta)$, $i=\overline{1,n}$. For fixed $k>0$ and $j=0,1,2,\dots$ set $\eta_{i,j} = \frac{1}{4}\eta_i(1+2^{-j})$, $i=\overline{1,n}$, $\eta_j = (\eta_{1,j},\eta_{2,j},\dots,\eta_{n,j})$, $k_j = k(1-2^{-j})$, $Q_j = Q_{\eta_j}(\bar{x})$, $A_{k_j,j} = \{x\in Q_j(\bar{x}) : F(u) > k_j\}$. Let $\zeta_j \in C_0^\infty(Q_j(\bar{x}))$, $0\le\zeta_j\le 1, \zeta_j = 1$ in $Q_{j+1}(\bar{x})$, $\left|\frac{\partial\zeta_j}{\partial x_i}\right| \le \gamma 2^j\eta_i^{-1}, i=\overline{1,n}$.

By Lemma 2.1 and the Hölder inequality we obtain

$$\int_{A_{k_{j+1},j+1}}(F(u)-k_{j+1})_+dx \le \left(\int_{A_{k_{j+1},j}}\left((F(u)-k_{j+1})_+\zeta_j^{p_n}\right)^{\frac{n}{n-1}}dx\right)^{\frac{n-1}{n}}\left|A_{k_{j+1},j+1}\right|^{\frac{1}{n}}$$
$$\le \gamma\prod_{i=1}^{n}\left(\int_{A_{k_{j+1},j}}\left|\frac{\partial}{\partial x_i}\left((F(u)-k_{j+1})_+\zeta_j^{p_n}\right)\right|dx\right)^{\frac{1}{n}}\left|A_{k_{j+1},j}\right|^{\frac{1}{n}}$$
$$\le \gamma\prod_{i=1}^{n}\left(\int_{A_{k_{j+1},j}} f(u)|u_{x_i}|^{p_i}\zeta^{p_n}dx\right)^{\frac{1}{np_i}}\left(\int_{A_{k_{j+1},j}} f(u)\zeta_j^{p_n}dx\right)^{\frac{p_i-1}{np_i}}\left|A_{k_{j+1},j}\right|^{\frac{1}{n}}$$
$$+\gamma\prod_{i=1}^{n}\left(\int_{A_{k_{j+1},j}}(F(u)-k_{j+1})_+\left|\frac{\partial\zeta_j}{\partial x_i}\right|\zeta_j^{p_n-1}dx\right)^{\frac{1}{n}}\left|A_{k_{j+1},j}\right|^{\frac{1}{n}}.$$

From this by Lemma 2.3 and by the evident inequality $(F(u)-k_j)_+ \ge \frac{k}{2^{j+1}}$ on $A_{k_{j+1},j}$, we obtain

$$y_{j+1} = \int_{A_{k_{j+1},j}}(F(u)-k_{j+1})_+dx$$
$$\le \gamma(1-\sigma)^{-\gamma}2^{j\gamma}k^{-\frac{1}{n}}\left(|Q_\eta(\bar{x})|^{-\frac{1}{n}} + k^{-\frac{p-1}{p}}\theta_n^{-p_n}\sum_{i=1}^{n}\delta^{p_i-1}(\theta)N^{p_n-p_i}(\theta)\right)y_j^{1+\frac{1}{n}}.$$

It follows from Lemma 2.2 that $y_j \to 0$ as $j\to\infty$, provided k is chosen to satisfy

$$k = \max\left(\theta_n^{-p_n}N^{p_n}(\theta)\sum_{i=1}^{n}\left(\frac{\delta(\theta)}{N(\theta)}\right)^{(p_i-1)\frac{p}{p-1}}, \gamma(1-\sigma)^{-\gamma}|Q_\eta(\bar{x})|^{-1}\int_{Q_{\frac{\eta}{2}}(\bar{x})}F(u)dx\right).$$

This implies that

$$F(u(\bar{x})) \le \gamma(1-\sigma)^{-\gamma}\theta_n^{-p_n}N^{p_n}(\theta)\sum_{i=1}^{n}\left(\frac{\delta(\theta)}{N(\theta)}\right)^{(p_i-1)\frac{p}{p-1}}dx$$
$$+\gamma(1-\sigma)^{-\gamma}|Q_\eta(\bar{x})|^{-1}\int_{Q_{\frac{\eta}{2}}(\bar{x})}F(u)dx. \tag{2.3}$$

Let $\xi \in C_0^\infty(Q_\eta(\bar{x}))$, $0 \le \xi \le 1, \xi = 1$ in $Q_{\frac{\eta}{2}}(\bar{x})$, $\left|\frac{\partial \xi}{\partial x_i}\right| \le \gamma \eta_i^{-1}, i = \overline{1,n}$. To estimate the integral on the right-hand side of (2.3) we test (1.3) by $\varphi = \xi^{p_n}$, using conditions (1.2) and the Hölder inequality we obtain

$$\int_{Q_{\frac{\eta}{2}}(\bar{x})} F(u)dx \le \delta(\theta) \int_{Q_\eta(\bar{x})} f(u)\xi^{p_n} dx$$

$$\le \gamma\delta(\theta) \sum_{i=1}^n \left(\sum_{j=1}^n \int_{Q_\eta(\bar{x})} |u_{x_j}|^{p_j} \xi^{p_n} dx \right)^{1-\frac{1}{p_i}} \left(\int_{Q_\eta(\bar{x})} |\xi_{x_i}|^{p_i} dx \right)^{\frac{1}{p_i}}. \tag{2.4}$$

Test (1.3) by $\varphi = u\xi^{p_n}$, using conditions (1.2) and the Young inequality we get

$$\sum_{j=1}^n \int_{Q_\eta(\bar{x})} |u_{x_j}|^{p_j} \xi^{p_n} dx \le \gamma \sum_{j=1}^n \int_{Q_\eta(\bar{x})} u^{p_j} |\xi_{x_j}|^{p_j} dx. \tag{2.5}$$

Combining (2.4), (2.5) we arrive at

$$\int_{Q_{\frac{\eta}{2}}(\bar{x})} F(u)dx \le \gamma(1-\sigma)^{-\gamma} \theta_n^{-p_n} \delta(\theta) N^{p_n - 1}(\theta) |Q_\eta(\bar{x})|. \tag{2.6}$$

Since $\bar{x}$ is an arbitrary point in $Q_{\sigma\eta}(x^{(o)})$, from (2.3), (2.6) we obtain the required (1.6), which proves Theorem 1.1.

2.4. *Proof of Proposition 1.1*

First note an inequality which is an immediate consequence of our choice of ψ

$$\psi(u)v \le \varepsilon^{-1}\psi(u)u + \psi(\varepsilon v)v, \quad \varepsilon, u, v > 0, \tag{2.7}$$

indeed if $v \le \varepsilon^{-1}u$, then $\psi(u)v \le \varepsilon^{-1}\psi(u)u$, and if $v \ge \varepsilon^{-1}u$, then $\psi(u)v \le \psi(\varepsilon v)v$, and in both cases (2.7) holds.

For $j = 0, 1, 2 \ldots$ define the sequences $\{\sigma_j\}, \{\theta_j\}, \{M_j\}$ by $\sigma_j := \frac{1-2^{-j-1}}{1-2^{-j-2}}$, $\theta_j := (\theta_{1j}, \theta_{2j}, \ldots, \theta_{nj})$, $\theta_{ij} = \theta_i \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^j}\right), i = \overline{1,n}$, $M_j := \sup_{Q_{\theta_j(x^{(o)})}} u$, if inequality (1.5) is violated, using (1.8) we write (1.6) for the pair of boxes $Q_{\theta_j}(x^{(o)})$ and $Q_{\theta_{j+1}}(x^{(o)})$. This gives

$$M_j \psi(M_j) \le \gamma 2^{j\gamma} \rho^{-\frac{p_n}{b}} M_{j+1}.$$

If $\varepsilon \in (0,1)$, then by (2.7) from the previous inequality we obtain

$$\psi(M_j) \le \psi(\varepsilon M_{j+1}) + \frac{1}{\varepsilon} \frac{\psi(M_j) M_j}{M_{j+1}} \le \psi(\varepsilon M_{j+1}) + \varepsilon^{-1} \gamma 2^{j\gamma} \rho^{-\frac{p_n}{b}}.$$

Using the condition (A) we arrive at recursive inequalities

$$\psi(M_j) \le \varepsilon^\mu \psi(M_{j+1}) + \varepsilon^{-1}\gamma 2^{j\gamma} \rho^{-\frac{p_n}{b}}, \quad j = 0, 1, 2 \ldots.$$

From this, by iteration

$$\psi(M_o) \le \varepsilon^{j\mu} \psi(M_j) + \varepsilon^{-1} \gamma \rho^{-\frac{p_n}{b}} \sum_{i=0}^{j-1} \varepsilon^{i\mu} 2^{i\gamma},$$

for every $j \ge 1$.

We choose $\varepsilon^\mu = 2^{-\gamma-1}$ so that the sum on the right-hand side can be majorized by a convergent series and let $j \to \infty$ to obtain

$$\psi(u(x^{(o)})) \le \psi(M_o) \le \gamma \rho^{-\frac{p_n}{b}}.$$

This proves Proposition 1.1. The proof of Proposition 1.2 is completely similar.

3. Keller–Osserman a priori sub-estimates for parabolic equations. Proof of Theorem 1.2 and Propositions 1.3 and 1.4

3.1. Local energy estimates

Let $(\bar{x},\bar{t}) \in \Omega_T$, for any $\eta_1,\dots,\eta_n > 0$, $\eta = (\eta_1,\dots,\eta_n)$ and $s > 0$ we define $Q_{\eta,s}(\bar{x},\bar{t}) := Q_\eta(\bar{x}) \times (\bar{t}-s,\bar{t}+s)$, if $Q_{\eta,s}(\bar{x},\bar{t}) \subset \Omega_T$ we let ζ denote a nonnegative piecewise smooth cutoff function vanishing on the parabolic boundary of $Q_{\eta,s}(\bar{x},\bar{t})$. We provide the proof of (1.18), assuming without loss that

$$2 < p_1 \le \dots \le p_{n-1} < p_n, \quad \min_{1\le i\le n} m_i > 1, \quad m_n(p_n-1) \le 1 + \frac{\kappa}{n}, \ p < n. \tag{3.1}$$

In what follows γ stands for a constant depending only on $n, \nu_1, \nu_2, p_1, \dots, p_n, m_1, \dots, m_n$ which may vary from line to line.

Lemma 3.1. *Let u be a nonnegative weak solution to (1.13) and let the conditions (1.14), (3.1) hold. Then for every cylinder $Q_{\eta,s}(\bar{x},\bar{t}) \subset \Omega_T$ and for every $k > 0$*

$$\begin{aligned}
&\sup_{|t-\bar{t}|<s} \int_{Q_\eta(\bar{x})} (\Phi(u)-k)_+^2 \zeta^{p_n} dx + \sum_{i=1}^n \iint_{A_{k,\eta,s}} g^2(u) u^{(m_i-1)(p_i-1)} |u_{x_i}|^{p_i} \zeta^{p_n} dxdt \\
&\quad + \iint_{A_{k,\eta,s}} f(u) g(u) (\Phi(u)-k)_+ \zeta^{p_n} dxdt \le \gamma \iint_{A_{k,\eta,s}} (\Phi(u)-k)_+^2 |\zeta_t| \zeta^{p_n-1} dxdt \\
&\quad + \gamma \sum_{i=1}^n \iint_{A_{k,\eta,s}} (\Phi(u)-k)_+^2 \delta^{p_i-2}(u) u^{(m_i-1)(p_i-1)} |\zeta_{x_i}|^{p_i} dxdt,
\end{aligned} \tag{3.2}$$

where $A_{k,\eta,s} = \{(x,t) \in Q_{\eta,s}(\bar{x},\bar{t}) : \Phi(u) > k\}$.

Proof. Testing identity (1.16) by $\varphi = (\Phi(u)-k)_+ g(u) \zeta^p$, using conditions (1.14), we obtain

$$\begin{aligned}
&\sup_{|t-\bar{t}|<s} \int_{Q_\eta(\bar{x})} (\Phi(u)-k)_+^2 \zeta^{p_n} dx + \iint_{A_{k,\eta,s}} f(u) g(u) (\Phi(u)-k)_+ \zeta^{p_n} dxdt \\
&\quad + \sum_{i=1}^n \iint_{A_{k,\eta,s}} \left(g^2(u) + g'(u)(\Phi(u)-k)_+\right) u^{(m_i-1)(p_i-1)} |u_{x_i}|^{p_i} \zeta^{p_n} dxdt \\
&\le \gamma \iint_{A_{k,\eta,s}} (\Phi(u)-k)_+^2 |\zeta_t| \zeta^{p_n-1} dxdt + \gamma \sum_{i=1}^n \iint_{A_{k,\eta,s}} \left(\sum_{j=1}^n g^2(u) u^{(m_j-1)(p_j-1)} |u_{x_j}|^{p_j} \zeta^{p_n} \right)^{1-\frac{1}{p_i}} \\
&\quad \times g^{\frac{2}{p_i}-1}(u) u^{(m_i-1)\frac{p_i-1}{p_i}} (\Phi(u)-k)_+ |\zeta_{x_i}| \zeta^{\frac{p_n}{p_i}-1} dxdt.
\end{aligned}$$

From this, using the Young inequality and the evident inequality $\frac{\Phi(u)}{g(u)} \le \delta(u)$ we arrive at the required (3.2). □

3.2. Proof of Theorem 1.2

Consider a cylinder $Q_{\theta,\tau}(x^{(0)},t^{(0)})$ and let $(\bar{x},\bar{t})$ be an arbitrary point in $Q_{\sigma\theta,\sigma\tau}(x^{(0)},t^{(0)})$. If $u(x^{(0)},t^{(0)}) \ge (\tau^{-1}\rho^{p_n})^{\frac{1}{m_n(p_n-1)-1}} + \sum_{i=1}^{n-1} \left(\theta_i^{-1} \rho^{\frac{p_n}{p_i}}\right)^{\frac{p_i}{m_n(p_n-1)-m_i(p_i-1)}}$, then $M(\theta,\tau) = \max(M(\theta,\tau),\delta(\theta,\tau)) \ge (\tau^{-1}\rho^{p_n})^{\frac{1}{m_n(p_n-1)-1}} + \sum_{i=1}^{n-1}\left(\theta_i^{-1}\rho^{\frac{p_n}{p_i}}\right)^{\frac{p_i}{m_n(p_n-1)-m_i(p_i-1)}}$, and hence $Q_{\eta,s}(\bar{x},\bar{t}) \subset Q_{\theta,\tau}(x^{(0)},t^{(0)})$, where $s = (1-\sigma)\theta_n^{p_n} M^{1-m_n(p_n-1)}(\theta,\tau)$, $\eta_i = (1-\sigma)\theta_n^{\frac{p_n}{p_i}} M^{m_i(p_i-1)-m_n(p_n-1)}(\theta,\tau)$, $i = \overline{1,n}$. For fixed $k > 0$ and

$l,j=0,1,2\dots$ set $\alpha_l=\frac{1}{4}(1+2^{-1}+\dots+2^{l})$, $\eta_{i,j,l}=(\alpha_l+\frac{1}{4}2^{-j-l-1})\eta_i$, $i=\overline{1,n}$, $\eta_{j,l}=(\eta_{1,j,l},\dots,\eta_{n,j,l})$, $s_{j,l}=(\alpha_l+\frac{1}{4}2^{-j-l-1})s$, $k_j=k(1-2^{-j})$, $Q_{j,l}=Q_{\eta_{j,l},s_{j,l}}(\bar{x},\bar{t})$, $A_{k_j,j,l}=\{(x,t)\in Q_{j,l}: F(u)>k_j\}$. Let $\zeta_j\in C_0^\infty(Q_{j,l})$, $0\le\zeta_j\le 1$, $\zeta_j=1$ in $Q_{j+1,l}$, $\left|\frac{\partial\zeta_j}{\partial x_i}\right|\le\gamma 2^{j+l-1}\eta_i$, $i=\overline{1,n}$, $\left|\frac{\partial\zeta_j}{\partial t}\right|\le\gamma 2^{j+l}s^{-1}$.

By the Hölder inequality and Lemma 2.1 we obtain

$$\iint_{A_{k_{j+1},j+1,l}}(\Phi(u)-k_{j+1})_+^2dxdt\le\left(\iint_{A_{k_{j+1},j+1,l}}\left((\Phi(u)-k_{j+1})_+^2\zeta_j^{p_n}\right)^{\frac{n+1}{n}}dxdt\right)^{\frac{n}{n+1}}|A_{k_{j+1},j+1,l}|^{\frac{1}{n+1}}\le$$
$$\le\gamma\left(\sup_{|t-\bar{t}|<s_{j,l}}\int_{Q_{\eta_j,l}(\bar{x})}(\Phi(u)-k_{j+1})_+^2\zeta_j^{p_n}dx\right)^{\frac{1}{n+1}}$$
$$\times\prod_{i=1}^{n}\left(\iint_{A_{k_{j+1},j,l}}\left|\left((\Phi(u)-k_{j+1})_+^2\zeta_j^{p_n}\right)_{x_i}\right|dxdt\right)^{\frac{n}{n+1}}|A_{k_{j+1},j,l}|^{\frac{1}{n+1}}. \tag{3.3}$$

Using the inequality $\Phi(u)-k_j\ge\frac{k}{2^{j+1}}$ on $A_{k_{j+1},j,l}$, $\frac{\Phi(u)}{g(u)}\le\delta(u)$, we estimate the second term on the right-hand side of (3.3) as follows

$$\iint_{A_{k_{j+1},j,l}}\left|\left((\Phi(u)-k_{j+1})_+^2\zeta_j^{p_n}\right)_{x_i}\right|dxdt\le\gamma\iint_{A_{k_{j+1},j,l}}g(u)(\Phi(u)-k_{j+1})_+|u_{x_i}|\zeta_j^{p_n}dxdt$$
$$+\gamma\iint_{A_{k_{j+1},j,l}}(\Phi(u)-k_{j+1})_+^2\left|\frac{\partial\zeta_j}{\partial x_i}\right|\zeta_j^{p_n-1}dxdt$$
$$\le\gamma 2^{j\gamma}k^{-\frac{p_i-1}{p_i}}\left(\iint_{A_{k_{j+1},j,l}}g^2(u)u^{(m_i-1)(p_i-1)}|u_{x_i}|^{p_i}\zeta_j^{p_n}dxdt\right)^{\frac{1}{p_i}}$$
$$\times\left(\iint_{A_{k_{j+1},j,l}}\left(\frac{\Phi(u)}{g(u)}\right)^{\frac{p_i}{p_i-1}}g(u)u^{m_n-m_i}f(u)(\Phi(u)-k_j)_+\zeta_j^{p_n}dxdt\right)^{\frac{p_i-1}{p_i}}$$
$$+\gamma\iint_{A_{k_j,j,l}}(\Phi(u)-k_j)_+^2\left|\frac{\partial\zeta_j}{\partial x_i}\right|\zeta_j^{p_n-1}dxdt$$
$$\le\gamma 2^{j\gamma}k^{-\frac{p_i-1}{p_i}}\delta(\theta,\tau)M^{\frac{m_n-m_i}{p_i}(p_i-1)}(\theta,\tau)\left(\iint_{A_{k_j,j,l}}g^2(u)u^{(m_i-1)(p_i-1)}|u_{x_i}|^{p_i}\zeta_j^{p_n}dxdt\right)^{\frac{1}{p_i}}$$
$$\times\left(\iint_{A_{k_j,j,l}}g(u)f(u)(\Phi(u)-k_j)_+\zeta_j^{p_n}dxdt\right)^{1-\frac{1}{p_i}}+\gamma\iint_{A_{k_j,j,l}}(\Phi(u)-k_j)_+^2\left|\frac{\partial\zeta_j}{\partial x_i}\right|dxdt. \tag{3.4}$$

Choosing k from the condition

$$k\ge\theta_n^{-p_n}\delta(\theta,\tau)M^{m_np_n-1}(\theta,\tau),$$

using Lemma 3.1, from (3.3), (3.4) we obtain

$$y_{j+1,l}=\iint_{A_{k_{j+1},j+1,l}}(\Phi(u)-k_{j+1})_+^2dxdt\le\gamma(1-\sigma)^{-\gamma}2^{(j+l)\gamma}k^{-\frac{2}{n+1}}\left|Q_{\eta,s}(\bar{x},\bar{t})\right|^{-\frac{1}{n+1}}y_{j,l}^{1+\frac{1}{n+1}}.$$

Let $Q_l=Q_{\alpha_l\eta,\alpha_l s}$, $\Phi_l=\sup_{Q_l}\Phi(u)$, it follows from Lemma 2.2 that $y_{j,l}\to 0$ as $j\to\infty$, provided k is chosen to satisfy

$$k^2=\gamma(1-\sigma)^{-\gamma}2^{\gamma l}\left|Q_{\eta,s}(\bar{x},\bar{t})\right|^{-1}\iint_{Q_{l+1}}\Phi^2(u)dxdt.$$

If $\varepsilon \in (0,1)$, then from the previous we obtain

$$\begin{aligned}\Phi_l &\le \gamma\theta_n^{-p_n}\delta(\theta,\tau)M^{m_n p_n-1}(\theta,\tau)\\ &\quad+\gamma(1-\sigma)^{-\gamma}2^{\gamma l}\delta^{\frac12}(\theta,\tau)M^{\frac{m_n-1}{2}}(\theta,\tau)\Phi_{l+1}^{\frac12}\left|Q_{\eta,s}(\bar x,\bar t)\right|^{-\frac12}\left(\iint_{Q_{\frac{\eta}{2},\frac{s}{2}}(\bar x,\bar t)} f(u)dxdt\right)^{\frac12}\\ &\le \varepsilon\Phi_{l+1}+\gamma\theta_n^{-p_n}\delta(\theta,\tau)M^{m_n p_n-1}(\theta,\tau)\\ &\quad+\gamma\varepsilon^{-1}(1-\sigma)^{-\gamma}2^{\gamma l}\delta(\theta,\tau)M^{m_n-1}(\theta,\tau)\left|Q_{\eta,s}(\bar x,\bar t)\right|^{-1}\iint_{Q_{\frac{\eta}{2},\frac{s}{2}}(\bar x,\bar t)} f(u)dxdt,\quad l=0,1,2\ldots.\end{aligned}$$

From this by iteration

$$\begin{aligned}\Phi(u(\bar x,\bar t)) &\le \Phi_0 \le \varepsilon^l\Phi_l+\gamma\varepsilon^{-1}\sigma^{-\gamma}\sum_{i=0}^{l-1}(\varepsilon 2^\gamma)^i\\ &\times\left(\theta_n^{-p_n}\delta(\theta,\tau)M^{m_n p_n-1}(\theta,\tau)+\delta(\theta,\tau)M^{m_n-1}(\theta,\tau)\left|Q_{\eta,s}(\bar x,\bar t)\right|^{-1}\iint_{Q_{\frac{\eta}{2},\frac{s}{2}}(\bar x,\bar t)} f(u)dxdt\right),\end{aligned}$$

for every $l \ge 1$.

Choosing $\varepsilon = 2^{-\gamma-1}$ so that the sum on the right-hand side can be majorized by a convergent series and $l \to \infty$ to obtain

$$\begin{aligned}\Phi(u(\bar x,\bar t)) &\le \gamma(1-\sigma)^{-\gamma}\theta_n^{-p_n}\delta(\theta,\tau)M^{m_n p_n-1}(\theta,\tau)dxdt\\ &\quad+\gamma(1-\sigma)^{-\gamma}\delta(\theta,\tau)M^{m_n-1}(\theta,\tau)\left|Q_{\eta,s}(\bar x,\bar t)\right|^{-1}\iint_{Q_{\frac{\eta}{2},\frac{s}{2}}(\bar x,\bar t)} f(u)dxdt.\end{aligned}\tag{3.5}$$

Let $\xi \in C_0^\infty(Q_{\eta,s}(\bar x,\bar t)), 0 \le \xi \le 1, \xi = 1$ in $Q_{\frac{\eta}{2},\frac{s}{2}}(\bar x,\bar t), \left|\frac{\partial\xi}{\partial x_i}\right| \le \gamma\eta_i^{-1},\ i=\overline{1,n}, \left|\frac{\partial\xi}{\partial t}\right| \le \gamma s^{-1}$. To estimate the integral on the right-hand side of (3.5) we test (1.17) by $\varphi = \frac{u}{u+\varepsilon}\xi^{p_n}$, using conditions (1.15) and the Hölder inequality, and passing $\varepsilon \to 0$ we obtain

$$\begin{aligned}\iint_{Q_{\eta,s}(\bar x,\bar t)} f(u)\xi^{p_n}dxdt &\le \gamma\iint_{Q_{\eta,s}(\bar x,\bar t)} u|\xi_t|\xi^{p_n-1}dxdt\\ &\quad+\gamma\sum_{i=1}^n\left(\sum_{j=1}^n\iint_{Q_{\eta,s}(\bar x,\bar t)}|u_{x_i}|^{p_j}\xi^{p_n}dxdt\right)^{1-\frac{1}{p_i}}\left(\iint_{Q_{\eta,s}(\bar x,\bar t)}|\xi_{x_i}|^{p_i}dxdt\right)^{\frac{1}{p_i}}.\end{aligned}$$

Testing (1.16) by $\varphi = u\xi^{p_n}$, using conditions (1.14) and the Young inequality we obtain

$$\iint_{Q_{\eta,s}(\bar x,\bar t)} f(u)\xi^{p_n}dxdt \le \gamma M(\theta,\tau)|Q_\eta(\bar x)|.\tag{3.6}$$

Combining (3.5), (3.6) we arrive at

$$\Phi(u(\bar x,\bar t)) \le \gamma\sigma^{-\gamma}\theta_n^{-p_n}\delta(\theta,\tau)M^{m_n p_n-1}(\theta,\tau).\tag{3.7}$$

Since $(\bar x,\bar t)$ is an arbitrary point in $Q_{\sigma\theta,\sigma\tau}(x^{(0)},t^{(0)})$ from (3.7) the required (1.18) follows. This proves Theorem 1.3. The proof of Propositions 1.3 and 1.4 is completely similar to that of Proposition 1.1

4. Harnack's inequality for elliptic equations. Proof of Theorem 1.3

Let $x^{(0)} \in \Omega$ and $B_{8\rho}(x^{(0)}) \subset \Omega$, fix $\bar x \in B_\rho(x^{(0)}),\ \sigma \in (0,1)$ and $0 < r \le \rho$, and let $\zeta \in C_0^\infty(B_r(\bar x)),\ 0 \le \zeta \le 1,\ \zeta = 1$ in $B_{\sigma\rho}(\bar x),\ |\nabla\zeta| \le (1-\sigma)^{-1}r^{-1}$.

Lemma 4.1. *Let all the conditions of Theorem 1.3 be fulfilled. Then for every* $0 < k < \sup_{B_{2\rho}(x^{(0)})} u$ *next inequalities hold*

$$\int_{B_r(\bar{x})} |\nabla (u-k)_+|^p \zeta^p dx \le \gamma (1-\sigma)^{-p} r^{-p} \|(u-k)_+\|^p_{L^\infty(B_r(\bar{x}))} |A^+_{k,r}|, \tag{4.1}$$

$$\int_{B_r(\bar{x})} |\nabla (k-u)_+|^p \zeta^p dx \le \gamma (1-\sigma)^{-p} r^{-p} k^p |A^-_{k,r}|, \tag{4.2}$$

where $A^{\pm}_{k,r} = B_r(\bar{x}) \bigcap \{(u-k)_\pm > 0\}$.

Proof. Testing (1.3) by $\varphi = (u-k)_+ \zeta^p$, using conditions (1.2) with $p = p_1 = \dots = p_n$ and the Young inequality we arrive at (4.1). To prove (4.2) we test identity (1.3) by $\varphi = (k-u)_+\zeta^p$, using conditions (1.2) with $p = p_1 = \dots = p_n$ and the Young inequality we obtain

$$\int_{B_r(\bar{x})} |\nabla (k-u)_+|^p \zeta^p dx \le \gamma (1-\sigma)^{-p} r^{-p} k^p |A^-_{k,r}| + \gamma \int_{B_r(\bar{x})} f(u)(k-u)_+ \zeta^p dx. \tag{4.3}$$

Let us estimate the integral on the right hand side of (4.3), we have

$$\int_{B_r(\bar{x})} f(u)(k-u)_+ \zeta^p dx = \int_{B_r(\bar{x})} f(u)\chi(u<k) \int_u^k ds\, \zeta^p\, dx$$
$$\le \int_{B_r(\bar{x})} \chi(u<k) \int_u^k f(s)ds\, \zeta^p dx \le \int_0^k f(s)ds\, |A^-_{k,r}| = F(k)|A^-_{k,r}|.$$

By Proposition 1.2 we get

$$\frac{F^{\frac{1}{p}}(k)}{k} \le \frac{F^{\frac{1}{p}}(\sup\limits_{B_{2\rho}(x^{(0)})} u)}{\sup\limits_{B_{2\rho}(x^{(0)})} u} \le \gamma \rho^{-1} \le \gamma r^{-1}$$

with constant γ independent of u, hence (4.3) yields (4.2). This proves the lemma. □

Lemma 4.1 implies that the solution u of Eq. (1.1) with $p = p_1 = \dots = p_n$ belongs to the corresponding elliptic B_p-class (see [8,19]) and hence u satisfies the Harnack inequality (1.25), for the details we refer the reader to [8].

5. Harnack's inequality for parabolic equations. Proof of Theorem 1.4

Unfortunately, due to the presence of the absorption term we cannot use the results from [6], since the B_p-classes considered in [6] are homogeneous. Hence, following the strategy [3], we give a sketch of the proof of the Harnack inequality. Note that in the case considered here the constant C from the structure conditions (1.2) in Chapter 3 of [3] is 0, and therefore, the first alternative $C_\rho > \min(1, u(x^{(0)}, t^{(0)}))$ from [3, Chapter 5, Theorem 1.1] will never occur.

5.1. Local energy estimates

Let $(x^{(0)}, t^{(0)})$ be an arbitrary point such that $u(x^{(0)}, t^{(0)}) > 0$ and for $r, \eta > 0$ construct the cylinders

$$Q_{r,\eta}(\bar{x}, \bar{t}) \subset Q_{2\rho, 2\theta}(x^{(0)}, t^{(0)}) \subset \Omega_T, \quad \theta = \rho^p \left(\frac{c_{14}}{u(x^{(0)}, t^{(0)})} \right)^{p-2}. \tag{5.1}$$

Let $\zeta \in C_0^\infty(Q_{r,\eta}(\bar{x}, \bar{t}))$, $0 \le \zeta \le 1$, $\zeta = 1$ in $Q_{\sigma r, \sigma\eta}(\bar{x}, \bar{t})$, $|\nabla \zeta| \le (1-\sigma)^{-1} r^{-1}$, $|\zeta_t| \le (1-\sigma)^{-1}\eta^{-1}$.

Lemma 5.1. *Let all the conditions of Theorem 1.1 be fulfilled, then for every* $0 < k < \sup_{Q_{2\rho,2\tau}(x^{(0)},t^{(0)})} u$

$$\sup_{\bar t-\eta<t<\bar t+\eta}\int_{B_r(\bar x)}(u-k)_+^2\zeta^p dx+\iint_{Q_{r,\eta}(\bar x,\bar t)}|\nabla(u-k)_+|^p\zeta^p dxdt$$
$$\le \gamma(1-\sigma)^{-1}\eta^{-1}\iint_{Q_{r,\eta}(\bar x,\bar t)}(u-k)_+^2 dxdt+\gamma(1-\sigma)^{-p}r^{-p}\iint_{Q_{r,\eta}(\bar x,\bar t)}(u-k)_+^p dxdt, \tag{5.2}$$
$$\sup_{\bar t-\eta<t<\bar t+\eta}\int_{B_r(\bar x)}(k-u)_+^2\zeta^p dx+\iint_{Q_{r,\eta}(\bar x,\bar t)}|\nabla(k-u)_+|^p\zeta^p dxdt$$
$$\le \gamma(1-\sigma)^{-1}\left(k^2\eta^{-1}+k^p\eta^{-p}\right)|A^-_{k,r,\eta}|, \tag{5.3}$$

where $A^-_{k,r,\eta}=Q_{r,\eta}(\bar x,\bar t)\bigcap\{u<k\}$.

Proof. Testing (1.16) by $\varphi=(u-k)_+\zeta^p$, using conditions (1.14) with $p=p_1=\cdots=p_n$, $m_1=\cdots=m_n=1$ and the Young inequality we arrive at (5.2). To prove (5.3) we test (1.16) by $\varphi=(k-u)_+\zeta^p$, using conditions (1.14) with $p=p_1=\cdots=p_n$, $m_1=\cdots=m_n=1$ and the Young inequality we obtain

$$\sup_{\bar t-\eta<t<\bar t+\eta}\int_{B_r(\bar x)}(k-u)_+^2\zeta^p dx+\iint_{Q_{r,\eta}(\bar x,\bar t)}|\nabla(k-u)_+|^p\zeta^p dx$$
$$\le \gamma(1-\sigma)^{-1}\eta^{-1}\iint_{Q_{r,\eta}(\bar x,\bar t)}(k-u)_+^2 dxdt+\gamma(1-\sigma)^{-p}r^{-p}\iint_{Q_{r,\eta}(\bar x,\bar t)}(k-u)_+^p dxdt$$
$$+\gamma\iint_{Q_{r,\eta}(\bar x,\bar t)}f(u)(k-u)_+\zeta^p dxdt.$$

Using Proposition 1.4 with $m_1=m_2=\cdots=m_n=1$, we estimate the integral on the right hand side of the previous inequality as follows

$$\iint_{Q_{r,\eta}(\bar x,\bar t)}f(u)(k-u)_+\zeta^p dxdt\le\gamma\Phi(k)|A^-_{k,r,\eta}|\le\gamma r^{-p}k^p|A^-_{k,r,\eta}|.$$

This proves (5.3). □

5.2. A De Giorgi—type lemma

The following lemma is a consequence of Lemmas 5.1, 2.2 and the embedding theorem (see [3, Chapter 3, Lemma 3.1]).

Lemma 5.2. *Let the conditions of Theorem 1.4 be fulfilled. Let* $(x^{(0)},t^{(0)})\in\Omega_T$ *be such that* $u(x^{(0)},t^{(0)})>0$, *fix* θ *as in (5.1),* $\xi,a\in(0,1)$, $0<\omega<\sup_{Q_{2\rho,2\theta}(x^{(0)},t^{(0)})}u$, $\theta^{-1}\le(\xi\omega)^{p-2}$. *There exists number* $\nu^-\in(0,1)$ *depending only on* ν_1,ν_2,k,p *and* a *such that if*

$$\left|\left\{(x,t)\in Q^-_{r,r^p(\xi\omega)^{2-p}}(\bar x,\bar t):u(x,t)\le\xi\omega\right\}\right|\le\nu^-\left|Q^-_{r,r^p(\xi\omega)^{2-p}}(\bar x,\bar t)\right|, \tag{5.4}$$

then

$$u(x,t)\ge a\xi\omega\quad \text{for a.a. } (x,t)\in Q^-_{\frac r2,\left(\frac r2\right)^p(\xi\omega)^{2-p}}(\bar x,\bar t), \tag{5.5}$$

for any $Q^-_{r,r^p(\xi\omega)^{2-p}}(\bar x,\bar t)=B_r(\bar x)\times\left(\bar t-r^p(\xi\omega)^{2-p},\bar t\right)\subset Q_{2\rho,2\theta}(x^{(0)},t^{(0)})\subset\Omega_T$. *Likewise, let* M *be some number satisfying the inequality* $M\ge\sup_{Q^-_{r,r^p(\xi\omega)^{2-p}}(\bar x,\bar t)}u$, *then there exists number* $\nu^+\in(0,1)$ *depending only on* ν_1,ν_2,n,p,a,M *and* ω *such that if*

$$\left|\left\{(x,t)\in Q^-_{r,r^p(\xi\omega)^{2-p}}(\bar x,\bar t):u(x,t)\ge M(1-\xi)\right\}\right|\le\nu^+\left|Q^-_{r,r^p(\xi\omega)^{2-p}}(\bar x,\bar t)\right|, \tag{5.6}$$

then

$$u(x,t) \le M(1-a\xi) \quad \textit{for a.a. } (x,t) \in Q^-_{\frac{r}{2},(\frac{r}{2})^p(\xi\omega)^{2-p}}(\bar{x},\bar{t}), \tag{5.7}$$

for any $Q^-_{r,r^p(\xi\omega)^{2-p}}(\bar{x},\bar{t}) \subset Q_{2\rho,2\theta}(x^{(0)},t^{(0)}) \subset \Omega_T$.

5.3. Expansion of positivity

The following lemma is an expansion of positivity result, analogue in formulation as well as in the proof to [3, Chapter 4, Proposition 4.1]. For $(\bar{x},\bar{t}) \in \Omega_T$ and some given $0 < N < \sup_{Q_{2\rho,2\theta}(x^{(0)},t^{(0)})} u$ consider the cylinder

$$B_{4r}(\bar{x}) \times \left(\bar{t}, \bar{t} + \frac{b^{p-2}}{(\varepsilon N)^{p-2}}\delta(4r)^p\right) \subset Q_{2\rho,2\theta}(x^{(0)},t^{(0)}) \subset \Omega_T,$$

where b, ε, δ are the positive constants given by Lemma 5.3.

Lemma 5.3. *Let the conditions of Theorem 1.4 be fulfilled. Assume that for some* $(\bar{x},\bar{t}) \in \Omega_T$*, some* $r > 0$ *and some* $\alpha \in (0,1)$

$$\left|\left\{x \in B_r(\bar{x}) : u(x,\bar{t}) < N\right\}\right| \le (1-\alpha)|B_r(\bar{x})|. \tag{5.8}$$

Then there exist constants $\varepsilon, \delta \in (0,1)$ *and* $b > 1$ *depending only on* ν_1, ν_2, p, n *and* α *such that*

$$u(x,t) \ge \varepsilon N \quad \textit{for a.a. } x \in B_{2r}(\bar{x}), \tag{5.9}$$

and for all times

$$\bar{t} + \frac{1}{2}\frac{b^{p-2}}{(\varepsilon N)^{p-2}}\delta r^p \le t \le \bar{t} + \frac{b^{p-2}}{(\varepsilon N)^{p-2}}\delta r^p \tag{5.10}$$

where $A^{\pm}_{k,r,\eta} = Q_{r,\eta}(\bar{x},\bar{t}) \bigcap \{(u-k)_\pm > 0\}$.

Proof. By (5.8), using (5.3) similar to [3, Chapter 4, Lemma 4.1] we obtain that for every $\tau > 0$

$$\left|\left\{x \in B_r(\bar{x}) : u(x, \bar{t} + e^\tau N^{2-p}\delta r^p) \le \varepsilon_1 e^{-\frac{\tau}{p-2}} N\right\}\right| \le \left(1 - \frac{\alpha}{2}\right)|B_r(\bar{x})|, \tag{5.11}$$

with some $\varepsilon_1, \delta \in (0,1)$ depending only on ν_1, ν_2, p, n and α.

In the same way as in [3, Chapter 2, Proposition 4.1] we consider the function

$$w(x,\tau) = e^{\frac{\tau}{p-2}} N^{-1}(\delta r^p)^{\frac{1}{p-2}} u(x, \bar{t} + e^\tau N^{2-p}\delta r^p), \quad \tau > 0.$$

Set $k_0 = \varepsilon_1(\delta r^p)^{\frac{1}{p-2}}$, inequality (5.11) translates into w as

$$|\{x \in B_r(\bar{x}) : w(x,\tau) \le k_0\}| \le \left(1 - \frac{\alpha}{2}\right)|B_r(\bar{x})|, \tag{5.12}$$

for every $\tau > 0$.

Since $w \ge 0$, formal differentiation, which can be justified in a standard way, gives

$$\begin{aligned} w_\tau &= \frac{1}{p-2}w + \left(e^{\frac{\tau}{p-2}} N^{-1}(\delta r^p)^{\frac{1}{p-2}}\right)^{p-1} u_t \\ &= \frac{1}{p-2}w + div\, \widetilde{A}(x,\tau,w,\nabla w) - \widetilde{a}_0(w), \end{aligned} \tag{5.13}$$

where $\widetilde{A}, \widetilde{a}_0$ satisfy the conditions

$$\begin{aligned} &\widetilde{A}(x,\tau,w,\nabla w)\nabla w \geq \nu_1 \sum_{i=1}^{n} |w_{x_i}|^p, \\ &|\widetilde{a}_i(x,\tau,w,\nabla w)| \leq \nu_2 \left(\sum_{i=1}^{n} |w_{x_i}|^p\right)^{1-\frac{1}{p}}, \quad 1=\overline{1,n}, \\ &\nu_1 \widetilde{f}(w) \leq \widetilde{a}_0(w) \leq \nu_2 \widetilde{f}(w), \end{aligned} \tag{5.14}$$

where $\widetilde{f}(w) = \left(e^{\frac{\tau}{p-2}} N^{-1}(\delta r^p)^{\frac{1}{p-2}}\right)^{p-1} f\left(we^{-\frac{\tau}{p-2}} N(\delta r^p)^{-\frac{1}{p-2}}\right)$.

Let $k_s = k_0 2^{-s}, s = 0,1,\ldots,s_*$, where s_* is a sufficiently large positive number, depending only on n, p, ν_1, ν_2, satisfying the condition $e^{2^{s_*(p-2)}} \leq c_{14}$. By our choices and by Proposition 1.4 we have

$$\begin{aligned} \widetilde{F}(k_s) &= \int_0^{k_s} \widetilde{f}(l)dl = \left(e^{\frac{\tau}{p-2}} N^{-1}(\delta r^p)^{\frac{1}{p-2}}\right)^{p} F\left(k_s e^{-\frac{\tau}{p-2}} N(\delta r^p)^{-\frac{1}{p-2}}\right) \\ &\leq \gamma k_s^p r^{-p}, \end{aligned}$$

for $x \in B_{4r}(\bar{x})$ and for all $0 < \tau \leq \ln c_{14}$.

Hence, the energy estimates (5.3) for the function $(k_s - w)_+$ over the cylinders $Q^+_{4r,\eta_*}(\bar{x},0) = B_{4r}(\bar{x}) \times (0,\eta_*)$, $\eta_* = k_{s_*}^{2-p} r^p$, can be written in the form

$$\sup_{0<\tau<\eta_*} \int_{B_{4r}(\bar{x})} (k_s - w)_+^2 \zeta^p dx + \iint_{Q^+_{4r,\eta_*}(\bar{x},0)} |\nabla (k_s - w)_+|^p \zeta^p dx d\tau \leq \gamma k_s^p r^{-p} |A^-_{k_s,4r,\eta_*}|,$$

where $A^-_{k_s,4r,\eta_*} = Q^+_{4r,\eta_*}(\bar{x},0) \bigcap \{w < \kappa_s\}$ and $\zeta \in C_0^\infty(Q^+_{4r,\eta_*}(\bar{x},0))$, $0 \leq \zeta \leq 1$, $\zeta = 1$ in $B_{2r}(\bar{x}) \times \left(\frac{\eta_*}{4}, \frac{\eta_*}{2}\right)$, $|\nabla \zeta| \leq \gamma r^{-1}$, $|\zeta_\tau| \leq \gamma \eta_*^{-1}$.

The rest of the proof of Lemma 5.3 is the same as in [3] (see [3, Chapter 4, Proposition 4.1] for details). □

After we have proved Lemmas 5.2 and 5.3 the rest of the arguments do not differ from [3] (see [3, Chapter 5, Theorem 1.1] for details). This completes the proof of Theorem 1.4.

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