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Research Article

Yuliya Gorban*

Existence of entropy solutions for nonlinear elliptic degenerate anisotropic equations

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Abstract: In the present article we deal with the Dirichlet problem for a class of degenerate anisotropic elliptic second-order equations with L^1-right-hand sides in a bounded domain of $\mathbb{R}^n$ ($n \geqslant 2$). This class is described by the presence of a set of exponents $q_1, \dots, q_n$ and a set of weighted functions $v_1, \dots, v_n$ in growth and coercitivity conditions on coefficients of the equations. The exponents q_i characterize the rates of growth of the coefficients with respect to the corresponding derivatives of unknown function, and the functions v_i characterize degeneration or singularity of the coefficients with respect to independent variables. Our aim is to investigate the existence of entropy solutions of the problem under consideration.

Keywords: Nonlinear elliptic degenerate anisotropic second-order equations, L^1-data, Dirichlet problem, Existence of entropy solutions

MSC: 35J25, 35J60, 35J70

1 Introduction

During the last twenty years the research on the existence and properties of solutions for nonlinear equations and variational inequalities with L^1-data or measures as data were intensively developed. As is generally known, an effective approach to the solvability of second-order equations in divergence form with L^1-right-hand sides has been proposed in [1]. In this connection we also mention a series of other close investigations for nondegenerate isotropic nonlinear second-order equations with L^1-data and measures, entropy and renormalized solutions [2–10].

As for the solvability of nonlinear elliptic second-order equations with anisotropy and degeneracy (with respect to the independent variables), we note the following works. The existence of a weak (distributional) solution to the Dirichlet problem for a model nondegenerate anisotropic equation with right-hand side measure was established in [11]. The existence of weak solutions for a class of nondegenerate anisotropic equations with locally integrable data in $\mathbb{R}^n$ ($n \geqslant 2$) was proved in [12], and an analogous existence result concerning the Dirichlet problem for a system of nondegenerate anisotropic equations with measure data was obtained in [13]. Moreover, in [14], the existence of weak solutions to the Dirichlet problem for nondegenerate anisotropic equations with right-hand sides from Lebesgues spaces close to L^1 was established. Solvability of the Dirichlet problem for degenerate isotropic equations with L^1-data and measures as data was studied in [15–19]. Remark that in [15, 17], the existence of entropy solutions to the given problem was proved in the case of L^1-data, and in [16], the existence of a renormalized solution of the problem for the same case was established. In [16, 18, 19], the existence of distributional solutions of the problem was obtained in the case of right-hand side measures.

Solvability of the Dirichlet problem for a class of degenerate anisotropic elliptic second-order equations with L^1-right-hand sides was studied in [20]. This class is described by the presence of a set of exponents $q_1, \dots, q_n$ and of a set of weighted functions $v_1, \dots, v_n$ in growth and coercitivity conditions on coefficients of the equations under

***Corresponding Author: Yuliya Gorban:** Donetsk National University, Vinnytsya, Ukraine, E-mail: yuliya_gorban@mail.ru

consideration. The exponents q_i characterize the rates of growth of the coefficients with respect to the corresponding derivatives of unknown function, and the functions v_i characterize degeneration or singularity of the coefficients with respect to the independent variables. This is the most general situation in comparison with the above-mentioned works: the nondegenerate isotropic case means that $v_i \equiv 1$ and $q_i = q_1, i = 1, \dots, n$; the nondegenerate anisotropic case means that $v_i \equiv 1, i = 1, \dots, n$, and $q_i, i = 1, \dots, n$, are generally different, and the degenerate isotropic case means that $v_i = v_1, i = 1, \dots, n$, as in [16–19] or $v_i, i = 1, \dots, n$, are generally different as in [15] but $q_i = q_1$, $i = 1, \dots, n$.

In [20], the theorem on the existence and uniqueness of entropy solution to the Dirichlet problem for this class of the equations was proved. Moreover, the existence results of some other types of solutions to the given problem were also obtained. Observe that the proofs of these theorems are based on use of some results of [21–23] on the existence and properties of solutions of second-order variational inequalities with L^1-right-hand sides and sufficiently general constraints. Note that in [20–23] right-hand sides to the investigated variational inequalities and equations depend on independent variables only, and belong to the class L^1.

The present article is devoted to the Dirichlet problem for a same class of the nonlinear elliptic second-order equations in divergence form with degenerate anisotropic coefficients as in [20]. Here right-hand sides to the given equations depend on independent variables and unknown function. A model example of this class is an equation

$$-\sum_{i=1}^{n} D_i(v_i(x)|D_i u|^{q_i-2}D_i u) = F(x,u), \quad x \in \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n \geqslant 2$), $1 < q_i < n$, $v_i > 0$ a.e. in Ω, $v_i \in L^1_{\mathrm{loc}}(\Omega)$, $(1/v_i)^{1/(q_i-1)} \in L^1(\Omega)$, $i = 1, \dots, n$, $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

The main result of this paper is a theorem on the existence of entropy solutions to the Dirichlet problem for the equations under consideration. We require an additional conditions to the function F in this theorem. Namely $F(x,u)$ has an arbitrary growth with respect to the second variable, and $F(x,u)$ belongs to $L^1(\Omega)$ under the fixed value of the second variable. In our case we have no opportunity to use the results [21–23] directly. We follow a general approach for proving the above-mentioned theorem. This approach has been proposed in [1] to the investigation on the existence and properties of solutions for nonlinear elliptic second-order equations with isotropic nondegenerate (with respect to the independent variables) coefficients and L^1-right-hand sides. In [21, 23] this approach has been taken to the anisotropic degenerate case. Also we use some ideas of [24].

2 Preliminaries

In this section we give some results of [23] which will be used in the sequel.

Let $n \in \mathbb{N}$, $n \geqslant 2$, Ω be a bounded domain in $\mathbb{R}^n$ with a boundary $\partial\Omega$, and for every $i \in \{1, \dots, n\}$ we have $q_i \in (1, n)$.

We set $q = \{q_i : i = 1, \dots, n\}$,

$$q_- = \min\{q_i : i = 1, \dots, n\}, \quad \overline{q} = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}\right)^{-1}, \quad \hat{q} = \frac{n(\overline{q}-1)}{(n-1)\overline{q}}.$$

Let for every $i \in \{1, \dots, n\}$ v_i be a nonnegative function on Ω such that $v_i > 0$ a.e. in Ω,

$$v_i \in L^1_{\mathrm{loc}}(\Omega), \quad (1/v_i)^{1/(q_i-1)} \in L^1(\Omega). \tag{1}$$

We set $v = \{v_i : i = 1, \dots, n\}$. We denote by $W^{1,q}(v,\Omega)$ the set of all functions $u \in W^{1,1}(\Omega)$ such that for every $i \in \{1, \dots, n\}$ we have $v_i|D_i u|^{q_i} \in L^1(\Omega)$.

Let $\|\cdot\|_{1,q,v}$ be the mapping from $W^{1,q}(v,\Omega)$ into $\mathbb{R}$ such that for every function $u \in W^{1,q}(v,\Omega)$

$$\|u\|_{1,q,v} = \int_\Omega |u|dx + \sum_{i=1}^{n}\left(\int_\Omega v_i|D_i u|^{q_i}dx\right)^{1/q_i}.$$

The mapping $\|\cdot\|_{1,q,\nu}$ is a norm in $W^{1,q}(\nu,\Omega)$, and, in view of the second inclusion of (1), the set $W^{1,q}(\nu,\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{1,q,\nu}$. Moreover, by virtue of the first inclusion of (1), we have $C_0^\infty(\Omega)\subset W^{1,q}(\nu,\Omega)$.

We denote by $\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in space $W^{1,q}(\nu,\Omega)$. Evidently, the set $\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$ is a Banach space with respect to the norm induced by the norm $\|\cdot\|_{1,q,\nu}$. It is obvious that $\overset{\circ}{W}{}^{1,q}(\nu,\Omega)\subset \overset{\circ}{W}{}^{1,1}(\Omega)$. Finally, we observe that $\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$ is a reflexive space. The proof of the latter statement can be found in [21].

Note that the following assertion hold.

Proposition 2.1. *If a sequence converges weakly in $\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$, then it converges strongly in $L^1(\Omega)$.*

Further, let for every $k>0$ $\quad T_k:\mathbb{R}\to\mathbb{R}$ be the function such that

$$T_k(s)=\begin{cases} s, & \text{if } |s|\leqslant k,\\ k\,\operatorname{sign} s, & \text{if } |s|>k.\end{cases}$$

By analogy with known results for nonweighted Sobolev spaces (see for instance [25]) we have: if $u\in\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$ and $k>0$, then $T_k(u)\in\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$ and for every $i\in\{1,\dots,n\}$

$$D_iT_k(u)=D_iu\cdot 1_{\{|u|<k\}}\quad \text{a. e. in}\quad \Omega. \tag{2}$$

We denote by $\overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$ the set of all functions $u:\Omega\to\mathbb{R}$ such that for every $k>0$, $T_k(u)\in\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$.

Clearly,

$$\overset{\circ}{W}{}^{1,q}(\nu,\Omega)\subset\overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega). \tag{3}$$

For every $u:\Omega\to\mathbb{R}$ and for every $x\in\Omega$ we set

$$k(u,x)=\min\{l\in\mathbb{N}:|u(x)|\leqslant l\}.$$

Definition 2.2. *Let $u\in\overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$ and $i\in\{1,\dots,n\}$. Then $\delta_iu:\Omega\to\mathbb{R}$ is the function such that for every $x\in\Omega$ $\quad\delta_iu(x)=D_iT_{k(u,x)}(u)\,(x)$.*

Definition 2.3. *If $u\in\overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$, then $\delta u:\Omega\to\mathbb{R}^n$ is the mapping such that for every $x\in\Omega$ and for every $i\in\{1,\dots,n\}$ $\quad(\delta u(x))_i=\delta_iu(x)$.*

Now we give several propositions which will be used in the next sections.

Proposition 2.4. *Let $u\in\overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$. Then for every $k>0$ we have $D_iT_k(u)=\delta_iu\cdot 1_{\{|u|<k\}}$ a.e. in Ω, $i=1,\dots,n$.*

If $u\in\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$, then for every $i\in\{1,\dots,n\}$ $\quad\delta_iu=D_iu$ a.e. in Ω.

Proposition 2.5. *Let $u\in\overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$ and $w\in\overset{\circ}{W}{}^{1,q}(\nu,\Omega)\cap L^\infty(\Omega)$. Then $u-w\in\overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$, and for every $i\in\{1,\dots,n\}$ and for every $k>0$ we have*

$$D_iT_k(u-w)=\delta_iu-D_iw\quad \text{a.e. in } \{|u-w|<k\}.$$

Proposition 2.6. *There exists a positive constant c_0 depending on n, q, and $\|1/\nu_i\|_{L^{1/(q_i-1)}(\Omega)}$, $i=1,\dots,n$, such that for every function $u\in\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$*

$$\left(\int_\Omega |u|^{n/(n-1)}dx\right)^{(n-1)/n}\leqslant c_0\prod_{i=1}^n\left(\int_\Omega \nu_i|D_iu|^{q_i}dx\right)^{1/nq_i}.$$

3 Statement of the Dirichlet problem. The concept of its entropy solution

Let $c_1, c_2 > 0$, $g_1, g_2 \in L^1(\Omega)$, $g_1, g_2 \geqslant 0$ in Ω, and let for every $i \in \{1, \dots, n\}$ $a_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function. We suppose that for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} (1/\nu_i)^{1/(q_i-1)}(x)|a_i(x,\xi)|^{q_i/(q_i-1)} \leqslant c_1 \sum_{i=1}^{n} \nu_i(x)|\xi_i|^{q_i} + g_1(x), \tag{4}$$

$$\sum_{i=1}^{n} a_i(x,\xi)\xi_i \geqslant c_2 \sum_{i=1}^{n} \nu_i(x)|\xi_i|^{q_i} - g_2(x). \tag{5}$$

Moreover, we assume that for almost every $x \in \Omega$ and for every $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$,

$$\sum_{i=1}^{n} [a_i(x,\xi) - a_i(x,\xi')]\,(\xi_i - \xi_i') > 0. \tag{6}$$

Now we give one result of [20] which will be used in the sequel.

Proposition 3.1. *The following assertions hold:*

a) *if* $u, w \in \overset{\circ}{W}{}^{1,q}(\nu, \Omega)$ *and* $i \in \{1, \dots, n\}$, *then* $a_i(x, \nabla u) D_i w \in L^1(\Omega)$;

b) *if* $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu, \Omega)$, $w \in \overset{\circ}{W}{}^{1,q}(\nu, \Omega) \cap L^\infty(\Omega)$, $k > 0$ *and* $i \in \{1, \dots, n\}$,

then $a_i(x, \delta u) D_i T_k(u - w) \in L^1(\Omega)$.

Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. We consider the following Dirichlet problem:

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, \nabla u) = F(x,u) \quad \text{in } \Omega, \tag{7}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{8}$$

Definition 3.2. *An entropy solution of problem* (7), (8) *is a function* $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu, \Omega)$ *such that:*

$$F(x,u) \in L^1(\Omega); \tag{9}$$

for every function $w \in \overset{\circ}{W}{}^{1,q}(\nu, \Omega) \cap L^\infty(\Omega)$ *and for every* $k \geqslant 1$

$$\int_\Omega \left\{ \sum_{i=1}^{n} a_i(x, \delta u) D_i T_k(u - w) \right\} dx \leqslant \int_\Omega F(x,u)\, T_k(u - w)\, dx. \tag{10}$$

Note that the left-hand integral in (10) is finite. It follows from assertion b) of Proposition 3.1. The right-hand integral in (10) is also finite. It follows from the boundedness of the function T_k and inclusion (9).

4 Main result

Next theorem is the main result of this paper.

Theorem 4.1. *Suppose the following conditions are satisfied:*

1) *for a.e.* $x \in \Omega$ *the function* $F(x, \cdot)$ *is nonincreasing on* $\mathbb{R}$;

2) *for any* $s \in \mathbb{R}$ *the function* $F(\cdot, s)$ *belongs to* $L^1(\Omega)$.

Then there exists an entropy solution of the Dirichlet problem (7), (8).

Proof. According to the approach from [1], we will consider a sequence of the approximating problems for the equations with smooth right-hand sides. Then we will obtain special estimates of the solutions of these problems. Finally, we will pass to the limit. The proof is in 9 steps.

Step 1. We set $f = F(\cdot, 0)$. Let for every $l \in \mathbb{N}$, $F_l : \Omega \times \mathbb{R} \to \mathbb{R}$ be the function such that

$$F_l(x, s) = T_l(f(x) - F(x, s)), \quad (x, s) \in \Omega \times \mathbb{R}.$$

By virtue of condition 1), we have:

$$\text{if } l \in \mathbb{N}, \text{ then for a.e. } x \in \Omega \text{ the function } F_l(x, \cdot) \text{ is nondecreasing on } \mathbb{R}. \tag{11}$$

Further, in view of condition 2), we have $f \in L^1(\Omega)$. Hence there exists $\{f_l\} \subset C_0^\infty(\Omega)$ such that:

$$\lim_{l \to \infty} \|f_l - f\|_{L^1(\Omega)} = 0, \tag{12}$$

$$\forall\, l \in \mathbb{N} \qquad \|f_l\|_{L^1(\Omega)} \leqslant \|f\|_{L^1(\Omega)}. \tag{13}$$

Using the inequalities (4)–(6), property (11), and well-known results on the solvability of the equations with monotone operators (see for instance [26]), we obtain: if $l \in \mathbb{N}$, then there exists the function $u_l \in \overset{\circ}{W}{}^{1,q}(v, \Omega)$ such that for every function $w \in \overset{\circ}{W}{}^{1,q}(v, \Omega)$

$$\int\limits_\Omega \left\{ \sum_{i=1}^n a_i(x, \nabla u_l) D_i w + F_l(x, u_l) w \right\} dx = \int\limits_\Omega f_l w \, dx. \tag{14}$$

It means that the function $u_l \in \overset{\circ}{W}{}^{1,q}(v, \Omega)$ is a generalized solution of the Dirichlet problem:

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i(x, \nabla u) + F_l(x, u) = f_l \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega.$$

We denote by c_i, $i = 3, 4, \ldots$, the positive constants depending only on n, q, c_1, c_2, $\|g_1\|_{L^1(\Omega)}$, $\|g_2\|_{L^1(\Omega)}$, $\|f\|_{L^1(\Omega)}$, $\|F(\cdot, -1)\|_{L^1(\Omega)}$, $\|F(\cdot, 1)\|_{L^1(\Omega)}$, $\|1/v_i\|_{L^{1/(q_i-1)}(\Omega)}$, $i = 1, \ldots, n$, and meas Ω.

Let us show that for every $k \geqslant 1$ and $l \in \mathbb{N}$ the following inequalities hold:

$$\int\limits_{\{|u_l| < k\}} \left\{ \sum_{i=1}^n v_i |D_i u_l|^{q_i} \right\} dx \leqslant c_3 k, \tag{15}$$

$$\int\limits_{\{|u_l| \geqslant k\}} |F_l(x, u_l)| \, dx \leqslant c_4. \tag{16}$$

In fact, let $k \geqslant 1$ and $l \in \mathbb{N}$. As $u_l \in \overset{\circ}{W}{}^{1,q}(v, \Omega)$, we have $T_k(u_l) \in \overset{\circ}{W}{}^{1,q}(v, \Omega)$. In view of (14) and (13) we obtain

$$\int\limits_\Omega \left\{ \sum_{i=1}^n a_i(x, \nabla u_l) D_i T_k(u_l) + F_l(x, u_l) T_k(u_l) \right\} dx \leqslant k \|f\|_{L^1(\Omega)}.$$

Using (2) and (5) in the left-hand side of this inequality, we get

$$c_2 \int\limits_{\{|u_l| < k\}} \left\{ \sum_{i=1}^n v_i |D_i u_l|^{q_i} \right\} dx + \int\limits_\Omega F_l(x, u_l) T_k(u_l) \, dx \leqslant k \|f\|_{L^1(\Omega)} + \|g_2\|_{L^1(\Omega)}. \tag{17}$$

Assertion (11) and properties of the function T_k imply that

$$F_l(x,u_l)T_k(u_l) \geqslant 0 \text{ a.e. in } \Omega, \tag{18}$$

$$F_l(x,u_l)T_k(u_l) = k|F_l(x,u_l)| \text{ a.e. in } \{|u_l| \geqslant k\}. \tag{19}$$

The estimate (15) follows from (18) and (17). Finally, the inequality (16) follows from (19) and (17).

Step 2. Now we show that for every $k \geqslant 1$ and $l \in \mathbb{N}$

$$\operatorname{meas}\{|u_l| \geqslant k\} \leqslant c_5 k^{-\hat{q}}, \tag{20}$$

$$\operatorname{meas}\{v_i^{1/q_i}|D_i u_l| \geqslant k\} \leqslant c_6 k^{-q_i\hat{q}/(1+\hat{q})}, \quad i = 1,\dots,n. \tag{21}$$

In fact, let $k \geqslant 1$ and $l \in \mathbb{N}$. We have $|T_k(u_l)| = k$ on $\{|u_l| \geqslant k\}$; then

$$k^{n/(n-1)}\operatorname{meas}\{|u_l| \geqslant k\} \leqslant \int_\Omega |T_k(u_l)|^{n/(n-1)}dx. \tag{22}$$

Using Proposition 2.6, (2) and (15), we obtain

$$\left(\int_\Omega |T_k(u_l)|^{n/(n-1)}dx\right)^{(n-1)/n} \leqslant c_0 \prod_{i=1}^n \left(\int_{\{|u_l|<k\}} v_i|D_i u_l|^{q_i}\,dx\right)^{1/nq_i} \leqslant c_0(c_3k)^{1/\overline{q}}.$$

The inequality (20) follows from the latter estimate and (22).

Next, we fix $i \in \{1,\dots,n\}$, and set

$$k_* = k^{q_i/(1+\hat{q})}, \quad G = \{|u_l| < k_*,\ v_i^{1/q_i}\,|D_i u_l| \geqslant k\}.$$

We have

$$\operatorname{meas}\{v_i^{1/q_i}\,|D_i u_l| \geqslant k\} \leqslant \operatorname{meas}\{|u_l| \geqslant k_*\} + \operatorname{meas} G. \tag{23}$$

From (20) it follows that

$$\operatorname{meas}\{|u_l| \geqslant k_*\} \leqslant c_5 k_*^{-\hat{q}}. \tag{24}$$

Moreover, in view of the set's G definition and (15) we get

$$k^{q_i}\operatorname{meas} G \leqslant \int_{\{|u_l|<k_*\}} v_i\,|D_i u_l|^{q_i}\,dx \leqslant c_3 k_*.$$

The inequality (21) follows from the latter estimate and (23), (24).

Step 3. Assertions (2) and (15) imply that for every $k \geqslant 1$ the sequence $\{T_k(u_l)\}$ is bounded in $\overset{\circ}{W}{}^{1,q}(v,\Omega)$. As the space $\overset{\circ}{W}{}^{1,q}(v,\Omega)$ is reflexive, then there exist an increasing sequence $\{l_h\} \subset \mathbb{N}$, and sequence $\{z_k\} \subset \overset{\circ}{W}{}^{1,q}(v,\Omega)$ such that for every $k \in \mathbb{N}$ we have a weak convergence $T_k(u_{l_h}) \to z_k$ in $\overset{\circ}{W}{}^{1,q}(v,\Omega)$. Without loss of generality it can be assumed that

$$\forall\, k \in \mathbb{N} \qquad T_k(u_l) \to z_k \ \text{ weakly in } \overset{\circ}{W}{}^{1,q}(v,\Omega). \tag{25}$$

Step 4. Let us show that the sequence $\{u_l\}$ is fundamental on measure.

Indeed, let $k \geqslant 1$, $l, j \in \mathbb{N}$. We fix $t > 0$, and set $G' = \{|u_l| < k, |u_j| < k, |u_l - u_j| \geqslant t\}$. It is clear that

$$\operatorname{meas}\{|u_l - u_j| \geqslant t\} \leqslant \operatorname{meas}\{|u_l| \geqslant k\} + \operatorname{meas}\{|u_j| \geqslant k\} + \operatorname{meas} G'. \tag{26}$$

As $t \leqslant |T_k(u_l) - T_k(u_j)|$ on G', we obtain

$$t\operatorname{meas} G' \leqslant \int_\Omega |T_k(u_l) - T_k(u_j)|\,dx.$$

This inequality, (20) and (26) imply that for every $k \geqslant 1$, and $l, j \in \mathbb{N}$

$$\operatorname{meas}\{|u_l - u_j| \geqslant t\} \leqslant 2c_5 k^{-\hat{q}} + t^{-1} \int\limits_{\Omega} |T_k(u_l) - T_k(u_j)|\, dx. \tag{27}$$

Let $\varepsilon > 0$. We fix $k \in \mathbb{N}$ such that

$$2c_5 k^{-\hat{q}} \leqslant \varepsilon/2. \tag{28}$$

Taking into account (25) and Proposition 2.1, we infer a strong convergence $T_k(u_l) \to z_k$ in $L^1(\Omega)$. Then there exists $N \in \mathbb{N}$ such that for every $l, j \in \mathbb{N}, l, j \geqslant N$

$$\int\limits_{\Omega} |T_k(u_l) - T_k(u_j)|\, dx \leqslant \varepsilon t/2.$$

From this inequality, (27), and (28) we deduce that for every $l, j \in \mathbb{N}, l, j \geqslant N$

$$\operatorname{meas}\{|u_l - u_j| \geqslant t\} \leqslant \varepsilon.$$

This means that the sequence $\{u_l\}$ is fundamental on measure.

Step 5. Now we show that for every $i \in \{1, \dots, n\}$ the sequence $\{v_i^{1/q_i} D_i u_l\}$ is fundamental on measure.

For every $t > 0$ and $l, j \in \mathbb{N}$ we put

$$N_t(l, j) = \operatorname{meas}\left\{ \sum_{i=1}^{n} v_i^{1/q_i} |D_i u_l - D_i u_j| \geqslant t \right\}.$$

Besides, for every $t > 0, h, k \geqslant 1$, and $l, j \in \mathbb{N}$ we set

$$E_{t,h,k}(l, j) = \left\{ \sum_{i=1}^{n} v_i^{1/q_i} |D_i u_l - D_i u_j| \geqslant t, \sum_{i=1}^{n} v_i^{1/q_i} |D_i u_l| \leqslant h, \sum_{i=1}^{n} v_i^{1/q_i} |D_i u_j| \leqslant h, |u_l - u_j| < \frac{1}{k} \right\}.$$

Using (21), we establish that for every $t > 0, h \geqslant n, k \geqslant 1$, and $l, j \in \mathbb{N}$

$$N_t(l, j) \leqslant 2c_6 n^{n+1} h^{-q_- \hat{q}/(1+\hat{q})} + \operatorname{meas}\{|u_l - u_j| \geqslant 1/k\} + \operatorname{meas} E_{t,h,k}(l, j). \tag{29}$$

Further, we get one estimate for some integrals over $E_{t,h,k}(l, j)$. So we introduce now auxiliary functions and sets.

Let for every $x \in \Omega$ $\Phi_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a function such that for every pair $(\xi, \xi') \in \mathbb{R}^n \times \mathbb{R}^n$

$$\Phi_x(\xi, \xi') = \sum_{i=1}^{n} [a_i(x, \xi) - a_i(x, \xi')] (\xi_i - \xi_i').$$

Recall that a_i, $i = 1, \dots, n$, are Carathéodory functions, and inequality (6) holds for almost every $x \in \Omega$ and every $\xi, \xi' \in \mathbb{R}^n, \xi \neq \xi'$. Then there exists a set $E \subset \Omega$, meas $E = 0$, such that:

(i) for every $x \in \Omega \backslash E$ the function Φ_x is continuous on $\mathbb{R}^n \times \mathbb{R}^n$;
(ii) for every $x \in \Omega \backslash E$ and $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$, we have $\Phi_x(\xi, \xi') > 0$.

Put for every $t > 0$, $h > t$, and $x \in \Omega$

$$G_{t,h}(x) = \left\{ (\xi, \xi') \in \mathbb{R}^n \times \mathbb{R}^n : \sum_{i=1}^{n} v_i^{1/q_i}(x) |\xi_i| \leqslant h, \sum_{i=1}^{n} v_i^{1/q_i}(x) |\xi_i'| \leqslant h, \sum_{i=1}^{n} v_i^{1/q_i}(x) |\xi_i - \xi_i'| \geqslant t \right\}.$$

As $v_i > 0$ a.e. in Ω, $i = 1, \dots, n$, then there exists a set $\tilde{E} \subset \Omega$, meas $\tilde{E} = 0$, such that the set $G_{t,h}(x)$ is nonempty for every $t > 0$, $h > t$, and $x \in \Omega \backslash \tilde{E}$.

Let for every $t > 0$ and $h > t$ $\mu_{t,h} : \Omega \to \mathbb{R}$ be a function such that

$$\mu_{t,h}(x) = \begin{cases} \min\limits_{G_{t,h}(x)} \Phi_x, & \text{if } x \in \Omega \backslash (E \cup \tilde{E}), \\ 0, & \text{if } x \in E \cup \tilde{E}. \end{cases}$$

For every $t > 0$ and $h > t$ we have $\mu_{t,h} > 0$ a.e. in Ω, and $\mu_{t,h} \in L^1(\Omega)$.

Let $t > 0, h \geqslant t+1, k \geqslant 1$, and $l, j \in \mathbb{N}$. We fix $x \in E_{t,h,k}(l,j) \backslash (E \cup \tilde{E})$, and set $\xi = \nabla u_l(x), \xi' = \nabla u_j(x)$. As $(\xi, \xi') \in G_{t,h}(x)$, then $\mu_{t,h}(x) \leqslant \Phi_x(\xi, \xi')$. This inequality and function's Φ_x definition imply that

$$\mu_{t,h}(x) \leqslant \sum_{i=1}^{n} [a_i(x, \nabla u_l(x)) - a_i(x, \nabla u_j(x))] \, (D_i u_l(x) - D_i u_j(x)).$$

Then, taking into account (6) and (2), we obtain

$$\int\limits_{E_{t,h,k}(l,j)} \mu_{t,h} \, dx \leqslant \int\limits_{\Omega} \left\{ \sum_{i=1}^{n} [a_i(x, \nabla u_l) - a_i(x, \nabla u_j)] D_i T_{1/k}(u_l - u_j) \right\} dx. \tag{30}$$

In view of (14) we have

$$\int\limits_{\Omega} \left\{ \sum_{i=1}^{n} a_i(x, \nabla u_l) D_i T_{1/k}(u_l - u_j) \right\} dx = \int\limits_{\Omega} f_l T_{1/k}(u_l - u_j) \, dx - \int\limits_{\Omega} F_l(x, u_l) T_{1/k}(u_l - u_j) \, dx,$$

$$\int\limits_{\Omega} \left\{ \sum_{i=1}^{n} a_i(x, \nabla u_j) D_i T_{1/k}(u_j - u_l) \right\} dx = \int\limits_{\Omega} f_j T_{1/k}(u_j - u_l) \, dx - \int\limits_{\Omega} F_j(x, u_j) T_{1/k}(u_j - u_l) \, dx.$$

From these equalities and (30) it follows that

$$\int\limits_{E_{t,h,k}(l,j)} \mu_{t,h} \, dx \leqslant \frac{1}{k} \int\limits_{\Omega} |f_l - f_j| \, dx + \frac{1}{k} \int\limits_{\Omega} |F_l(x, u_l) - F_j(x, u_j)| \, dx. \tag{31}$$

Using (16) and conditions 1), 2), we find that for every $l, j \in \mathbb{N}$

$$\int\limits_{\Omega} |F_l(x, u_l) - F_j(x, u_j)| \, dx \leqslant c_7.$$

From the latter estimate and (31) we deduce that for every $t > 0$, $h \geqslant t + 1$, $k \geqslant 1$, and $l, j \in \mathbb{N}$ the following inequality holds:

$$\int\limits_{E_{t,h,k}(l,j)} \mu_{t,h} \, dx \leqslant \frac{1}{k} \int\limits_{\Omega} |f_l - f_j| \, dx + \frac{c_7}{k}. \tag{32}$$

The sequence $\{u_l\}$ is fundamental on measure. Then there exists an increasing sequence $\{n_k\} \subset \mathbb{N}$ such that for every $k \in \mathbb{N}$ and $l, j \in \mathbb{N}, l, j \geqslant n_k$, we have

$$\operatorname{meas} \{|u_l - u_j| \geqslant 1/k\} \leqslant 1/k. \tag{33}$$

Let $t > 0$ and $\varepsilon > 0$. We fix $h \geqslant t + n$ such that

$$2c_6 n^{n+1} h^{-q_- \hat{q}/(1+\hat{q})} \leqslant \varepsilon/4. \tag{34}$$

Put for every $k \in \mathbb{N}$

$$\alpha_k = \sup_{l, j \geqslant n_k} \operatorname{meas} E_{t,h,k}(l, j).$$

Let us show that $\alpha_k \to 0$. Assume the converse. Then there exist $\tau > 0$, an increasing sequence $\{k_s\} \subset \mathbb{N}$, and sequences $\{l_s\}, \{j_s\} \subset \mathbb{N}$ such that for every $s \in \mathbb{N}$ we have $l_s, j_s \geqslant n_{k_s}$ and

$$\operatorname{meas} E_{t,h,k_s}(l_s, j_s) \geqslant \tau. \tag{35}$$

Let $G_s = E_{t,h,k_s}(l_s, j_s)$, $s \in \mathbb{N}$.

In view of (32) and (13) for every $s \in \mathbb{N}$ we get

$$\int\limits_{G_s} \mu_{t,h}\, dx \leqslant \frac{c_7 + 2}{k_s}.$$

It follows that

$$\lim_{s\to\infty} \int\limits_{G_s} \mu_{t,h}\, dx = 0.$$

From this assertion, taking into account $\mu_{t,h} \in L^1(\Omega)$ and $\mu_{t,h} > 0$ a.e. in Ω, we infer that meas $G_s \to 0$. This fact is in contradiction to (35). Hence, we conclude that $\alpha_k \to 0$.

Finally, we fix $k \in \mathbb{N}$ such that the inequalities hold:

$$1/k \leqslant \varepsilon/4, \qquad \alpha_k \leqslant \varepsilon/2. \tag{36}$$

Let $l, j \in \mathbb{N}$, $l, j \geqslant n_k$. From (29), (33), (34) and (36) it follows that $N_t(l, j) \leqslant \varepsilon$.

This means that for every $i \in \{1, \dots, n\}$ the sequence $\{v_i^{1/q_i} D_i u_l\}$ is fundamental on measure.

Step 6. From results of the Steps 4 and 5, and F. Riesz's theorem we get the following facts: there exist measurable functions $u : \Omega \to \mathbb{R}$ and $v^{(i)} : \Omega \to \mathbb{R}$, $i = 1, \dots, n$, such that the sequence $\{u_l\}$ converges to u on measure, and for every $i \in \{1, \dots, n\}$ the sequence $\{v_i^{1/q_i} D_i u_l\}$ converges to $v^{(i)}$ on measure. As is generally known, we can extract subsequences converging almost everywhere in Ω to the corresponding functions. We may assume without loss of generality that

$$u_l \to u \ \text{ a.e. in } \Omega, \tag{37}$$

$$\forall\, i \in \{1, \dots, n\} \quad v_i^{1/q_i} D_i u_l \to v^{(i)} \ \text{ a.e. in } \Omega. \tag{38}$$

From (37), (25) and Proposition 2.1 we deduce that for every $k \in \mathbb{N}$

$$T_k(u) \in \overset{\circ}{W}{}^{1,q}(v, \Omega), \tag{39}$$

$$T_k(u_l) \to T_k(u) \ \text{ weakly in } \overset{\circ}{W}{}^{1,q}(v, \Omega). \tag{40}$$

Let us show that $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(v, \Omega)$. Indeed, let $k > 0$. Take $h \in \mathbb{N}$, $h > k$. In view of (39) we have $T_h(u) \in \overset{\circ}{W}{}^{1,q}(v, \Omega)$. Hence, by inclusion (3) we obtain $T_k(T_h(u)) \in \overset{\circ}{W}{}^{1,q}(v, \Omega)$. This fact and the equality $T_k(u) = T_k(T_h(u))$ imply that $T_k(u) \in \overset{\circ}{W}{}^{1,q}(v, \Omega)$. Therefore, $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(v, \Omega)$.

Step 7. Now we show that

$$\forall\, i \in \{1, \dots, n\} \qquad D_i u_l \to \delta_i u \ \text{ a.e. in } \Omega. \tag{41}$$

In fact, let $i \in \{1, \dots, n\}$. In view of (37) there exists a set $E' \subset \Omega$, meas $E' = 0$, such that

$$\forall\, x \in \Omega \backslash E' \qquad u_l(x) \to u(x), \tag{42}$$

and in view of (38) there exists a set $E'' \subset \Omega$, meas $E'' = 0$, such that

$$\forall\, x \in \Omega \backslash E'' \qquad v_i^{1/q_i}(x) D_i u_l(x) \to v^{(i)}(x). \tag{43}$$

Fix $k \in \mathbb{N}$. By (2) we have: if $l \in \mathbb{N}$, then there exists a set $E^{(l)} \subset \Omega$, meas $E^{(l)} = 0$, such that

$$\forall\, x \in \{|u_l| < k\} \backslash E^{(l)} \qquad D_i T_k(u_l)(x) = D_i u_l(x). \tag{44}$$

We denote by $\hat{E}$ a union of sets E', E'' and $E^{(l)}$, $l \in \mathbb{N}$. Clearly, meas $\hat{E} = 0$. Let $x \in \{|u| < k\} \backslash \hat{E}$. In view of (42) there exists $l_0 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant l_0$, we have $|u_l(x)| < k$. Let $l \in \mathbb{N}$, $l \geqslant l_0$. Then

$x \in \{|u_l| < k\} \setminus E^{(l)}$ and according to (44) we get $v_i^{1/q_i}(x) D_i T_k(u_l)(x) = v_i^{1/q_i}(x) D_i u_l(x)$. From this equality and (43) we deduce that $v_i^{1/q_i} D_i T_k(u_l)(x) \to v^{(i)}(x)$. Thus,

$$v_i^{1/q_i} D_i T_k(u_l) \to v^{(i)} \qquad \text{a.e. in } \{|u| < k\}. \tag{45}$$

Besides, in view of (2) and (15) for every $l \in \mathbb{N}$

$$\int_\Omega v_i |D_i T_k(u_l)|^{q_i} \, dx \leqslant c_3 k. \tag{46}$$

Using Fatou's lemma, from (45) and (46) we infer that the function $|v^{(i)}|^{q_i}$ is summable in $\{|u| < k\}$.

Further, let $\varphi : \Omega \to \mathbb{R}$ be a measurable function such that $|\varphi| \leqslant 1$ in Ω, and let $\varepsilon > 0$. As the function $|v^{(i)}|$ is summable on $\{|u| < k\}$, then there exists $\varepsilon_1 \in (0, \varepsilon)$ such that for every measurable set $G \subset \{|u| < k\}$, $\operatorname{meas} G \leqslant \varepsilon_1$, we have

$$\int_G |v^{(i)}| \, dx \leqslant \varepsilon. \tag{47}$$

Moreover, in view of (45) and Egorov's theorem, there exists a measurable set $\Omega' \subset \{|u| < k\}$ such that

$$\operatorname{meas}(\{|u| < k\} \setminus \Omega') \leqslant \varepsilon_1, \tag{48}$$

$$v_i^{1/q_i} D_i T_k(u_l) \to v^{(i)} \text{ uniformly in } \Omega'. \tag{49}$$

From (47) and (48) we infer that

$$\int_{\{|u|<k\}\setminus\Omega'} |v^{(i)}| \, dx \leqslant \varepsilon, \tag{50}$$

and from (49) we deduce that there exists $l_1 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant l_1$,

$$\int_{\Omega'} |v_i^{1/q_i} D_i T_k(u_l) - v^{(i)}| \, dx \leqslant \varepsilon. \tag{51}$$

Let $l \in \mathbb{N}$, $l \geqslant l_1$. Using (50), (51), Hölder's inequality, (48), and (46), we get

$$\begin{aligned}\left| \int_{\{|u|<k\}} [v_i^{1/q_i} D_i T_k(u_l) - v^{(i)}] \varphi \, dx \right| &\leqslant 2\varepsilon + \int_{\{|u|<k\}\setminus\Omega'} v_i^{1/q_i} |D_i T_k(u_l)| \, dx \\ &\leqslant 2\varepsilon + \varepsilon^{(q_i-1)/q_i} \left(\int_\Omega v_i |D_i T_k(u_l)|^{q_i} \, dx \right)^{1/q_i} \leqslant 2\varepsilon + \varepsilon^{(q_i-1)/q_i} (c_3 k)^{1/q_i}.\end{aligned}$$

Since ε is an arbitrary constant, from the latter estimate it follows that

$$\lim_{l\to\infty} \int_{\{|u|<k\}} [v_i^{1/q_i} D_i T_k(u_l) - v^{(i)}] \varphi \, dx = 0. \tag{52}$$

On the other hand, let $F : \overset{\circ}{W}{}^{1,q}(v, \Omega) \to \mathbb{R}$ be a functional such that for every function $v \in \overset{\circ}{W}{}^{1,q}(v, \Omega)$

$$\langle F, v \rangle = \int_{\{|u|<k\}} v_i^{1/q_i} D_i v \cdot \varphi \, dx.$$

It is easy to see that $F \in (\overset{\circ}{W}{}^{1,q}(v, \Omega))^*$. Hence, by virtue of (40), we have

$$\langle F, T_k(u_l) \rangle \to \langle F, T_k(u) \rangle.$$

This fact and functional's F definition imply that

$$\lim_{l\to\infty}\int\limits_{\{|u|<k\}} v_i^{1/q_i}\,D_i T_k(u_l)\cdot\varphi\,dx=\int\limits_{\{|u|<k\}} v_i^{1/q_i}\,D_i T_k(u)\cdot\varphi\,dx. \tag{53}$$

From (52) and (53) we deduce that

$$\int\limits_{\{|u|<k\}}[\,v^{(i)}-v_i^{1/q_i}\,D_i T_k(u)]\,\varphi\,dx=0.$$

In turn, from this equality and Proposition 2.1 we infer that

$$v^{(i)}=v_i^{1/q_i}\delta_i u\quad\text{a.e. in}\quad\{|u|<k\}.$$

Since $k\in\mathbb{N}$ is an arbitrary number, from the latter assertion it follows that

$$v^{(i)}=v_i^{1/q_i}\delta_i u\quad\text{a.e. in }\ \Omega. \tag{54}$$

Taking into account that $v_i>0$ a.e. in Ω, from (38) and (54) we obtain that $D_i u_l\to\delta_i u$ a.e. in Ω. Thus, (41) is proved.

Assertion (41) along with the fact that $a_i,\ i=1,\dots,n$, are Carathéodory functions implies that

$$\forall\, i\in\{1,\dots,n\}\qquad a_i(x,\nabla u_l)\to a_i(x,\delta u)\quad\text{a.e. in}\quad\Omega. \tag{55}$$

Step 8. Let us show that the following assertions are fulfilled:

$$F(x,u)\in L^1(\Omega); \tag{56}$$

$$F_l(x,u_l)\to f-F(x,u)\text{ strongly in }L^1(\Omega). \tag{57}$$

Indeed, in view of (37) we have

$$F_l(x,u_l)\to f-F(x,u)\text{ a.e. in }\Omega. \tag{58}$$

Moreover, using (16) and conditions 1) and 2), we get for every $l\in\mathbb{N}$

$$\int\limits_{\Omega}|F_l(x,u_l)|\,dx\leqslant c_8.$$

From this fact, (58), and Fatou's lemma we obtain inclusion (56).

Now let us prove (57). Firstly, we establish that for every $k,\,l\in\mathbb{N}$ the following estimate holds

$$\int\limits_{\{|u_l|\geqslant 2k\}}|F_l(x,u_l)|\,dx\leqslant\int\limits_{\{|u_l|\geqslant k\}}|f|\,dx+\|f_l-f\|_{L^1(\Omega)}+2\|g_2\|_{L^1(\Omega)}k^{-1}. \tag{59}$$

Let $z\in C^1(\mathbb{R})$ be a function such that $0\leqslant z\leqslant 1$ on $\mathbb{R}$, $z=0$ on $[-1;1]$, $z=1$ on $(-\infty;-2]\cup[2;+\infty)$, and for every $s\in\mathbb{R}\quad z'(s)\operatorname{sign}s\geqslant 0$, $|z'(s)|\leqslant 2$.

We fix arbitrary $k,\,l\in\mathbb{N}$. We denote by $z_k:\mathbb{R}\to\mathbb{R}$ a function such that for every $s\in\mathbb{R}$

$$z_k(s)=T_1\left(\frac{s}{k}\right)z\left(\frac{s}{k}\right). \tag{60}$$

From the properties of the functions T_1 and z it follows that for every $s\in\mathbb{R}$

$$|z_k(s)|\leqslant 1. \tag{61}$$

Besides,

$$\forall s\in\mathbb{R},\ |s|\leqslant k,\quad z_k(s)=0; \tag{62}$$

$$\forall s \in \mathbb{R},\ |s| \geqslant 2k, \quad |z_k(s)| = 1. \tag{63}$$

Definition (60) implies that $z_k(u_l) \in \overset{\circ}{W}{}^{1,q}(v, \Omega)$ and

$$D_i z_k(u_l) = k^{-1} z'\left(\frac{u_l}{k}\right) T_1\left(\frac{u_l}{k}\right) D_i u_l \quad \text{a.e. in } \Omega, \quad i = 1, \ldots, n. \tag{64}$$

Substituting $w = z_k(u_l)$ into (14), and using (61), (62), we get

$$\int\limits_{\Omega} \left\{ \sum_{i=1}^{n} a_i(x, \nabla u_l) D_i z_k(u_l) \right\} dx + \int\limits_{\Omega} F_l(x, u_l) z_k(u_l)\, dx \leqslant \int\limits_{\{|u_l| \geqslant k\}} |f|\, dx + \|f_l - f\|_{L^1(\Omega)}. \tag{65}$$

We denote by $I'_{k,l}$ the first integral in the left-hand side of (65). In view of the function's z definition

$$\forall s \in \mathbb{R},\ |s| \leqslant k, \text{ or } |s| \geqslant 2k, \quad |z'(s)| = 0. \tag{66}$$

Using (64), (66), and (5), we establish that

$$I'_{k,l} = k^{-1} \int\limits_{\{k \leqslant |u_l| \leqslant 2k\}} \left[z'\left(\frac{u_l}{k}\right) T_1\left(\frac{u_l}{k}\right) \left\{ \sum_{i=1}^{n} a_i(x, \nabla u_l) D_i u_l \right\} \right] dx$$

$$\geqslant k^{-1} \int\limits_{\{k \leqslant |u_l| \leqslant 2k\}} \left[z'\left(\frac{u_l}{k}\right) T_1\left(\frac{u_l}{k}\right) \left\{ c_2 \sum_{i=1}^{n} v_i |D_i u_l|^{q_i} - g_2 \right\} \right] dx. \tag{67}$$

From the truncated function's property and our condition $z'(s)\operatorname{sign} s \geqslant 0$, $\forall s \in \mathbb{R}$, it follows that almost everywhere in $\{k \leqslant |u_l| \leqslant 2k\}$

$$z'\left(\frac{u_l}{k}\right) T_1\left(\frac{u_l}{k}\right) = z'\left(\frac{u_l}{k}\right) \operatorname{sign}\left(\frac{u_l}{k}\right) \geqslant 0.$$

Taking into account this fact and our condition $|z'(s)| \leqslant 2$, $\forall s \in \mathbb{R}$, we deduce from (67)

$$I'_{k,l} \geqslant -2k^{-1} \int\limits_{\{k \leqslant |u_l| \leqslant 2k\}} g_2\, dx.$$

This and (65) imply

$$\int\limits_{\Omega} F_l(x, u_l) z_k(u_l)\, dx \leqslant \int\limits_{\{|u_l| \geqslant k\}} |f|\, dx + \|f_l - f\|_{L^1(\Omega)} + 2\|g_2\|_{L^1(\Omega)} k^{-1}. \tag{68}$$

Note that in view of (11) and the function's z_k definition we have

$$F_l(x, u_l) z_k(u_l) \geqslant 0 \text{ a.e. in } \Omega,$$

and in view of (63) we get

$$F_l(x, u_l) z_k(u_l) = |F_l(x, u_l)| \text{ a.e. in } \{|u_l| \geqslant 2k\}.$$

Then

$$\int\limits_{\Omega} F_l(x, u_l) z_k(u_l)\, dx \geqslant \int\limits_{\{|u_l| \geqslant 2k\}} |F_l(x, u_l)|\, dx.$$

Finally, assertion (59) is derived from the latter inequality and (68).

Next, we fix an arbitrary $\varepsilon > 0$. It is clear that there exists $\varepsilon_1 > 0$ such that for every measurable set $G \subset \Omega$, meas $G \leqslant \varepsilon_1$,

$$\int\limits_{G} (|f| + |F(x, u)|)\, dx \leqslant \varepsilon.$$

We fix $k \in \mathbb{N}$ such that the following inequalities hold:

$$2\|g_2\|_{L^1(\Omega)}k^{-1} \leqslant \varepsilon, \tag{69}$$

$$c_5 k^{-\hat{q}} \leqslant \varepsilon_1. \tag{70}$$

By condition 2), we infer that the functions $F(\cdot, -2k)$ and $F(\cdot, 2k)$ belong to $L^1(\Omega)$. Hence, there exists $\varepsilon_2 > 0$ such that for every measurable set $G \subset \Omega$, meas $G \leqslant \varepsilon_2$,

$$\int\limits_G (|F(\cdot, -2k)| + |F(\cdot, 2k)|)\, dx \leqslant \varepsilon.$$

In view of (58) there exists a measurable set $\Omega_1 \subset \Omega$ such that

$$\text{meas } (\Omega \setminus \Omega_1) \leqslant \min(\varepsilon_1, \varepsilon_2), \tag{71}$$

and $F_l(x, u_l) \to f - F(x, u)$ uniformly in Ω_1. Then there exists $L_1 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant L_1$,

$$\int\limits_{\Omega_1} |F_l(x, u_l) - (f - F(x, u))|\, dx \leqslant \varepsilon. \tag{72}$$

Besides, in view of (12) there exists $L_2 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant L_2$,

$$\|f_l - f\|_{L^1(\Omega)} \leqslant \varepsilon. \tag{73}$$

Now fix $l \in \mathbb{N}$, $l \geqslant \max(L_1, L_2)$. Using (71) and (72), we obtain

$$\|F_l(x, u_l) - (f - F(x, u))\|_{L^1(\Omega)} \leqslant \int\limits_{\{|u_l| \geqslant 2k\}} |F_l(x, u_l)|\, dx + \int\limits_{(\Omega\setminus\Omega_1)\cap\{|u_l|<2k\}} |F_l(x, u_l)|\, dx + 2\varepsilon. \tag{74}$$

By virtue of (20) and (70) we get meas $\{|u_l| \geqslant k\} \leqslant \varepsilon_1$. Then

$$\int\limits_{\{|u_l| \geqslant k\}} |f|\, dx \leqslant \varepsilon. \tag{75}$$

From (59), (69), (73) and (75) we deduce

$$\int\limits_{\{|u_l| \geqslant 2k\}} |F_l(x, u_l)|\, dx \leqslant 3\varepsilon. \tag{76}$$

Note that by condition 1), we have

$$|F_l(x, u_l)| \leqslant 2(|F(x, -2k)| + |F(x, 2k)|) \text{ a.e. in } \{|u_l| < 2k\}. \tag{77}$$

In view of (71) we get

$$\int\limits_{\Omega\setminus\Omega_1} (|F(x, -2k)| + |F(x, 2k)|)\, dx \leqslant \varepsilon. \tag{78}$$

From (77) and (78) it follows that

$$\int\limits_{(\Omega\setminus\Omega_1)\cap\{|u_l|<2k\}} |F_l(x, u_l)|\, dx \leqslant 2\varepsilon. \tag{79}$$

Using (74), (76), and (79), we infer

$$\|F_l(x, u_l) - (f - F(x, u))\|_{L^1(\Omega)} \leqslant 5\varepsilon.$$

Now we can conclude that $\|F_l(x,u_l)-(f-F(x,u))\|_{L^1(\Omega)}\to 0$. Thus, assertion (57) is proved.

Step 9. Let $w\in \overset{\circ}{W}{}^{1,q}(v,\Omega)\cap L^\infty(\Omega)$, and $k\geqslant 1$. Now we show that

$$\int\limits_\Omega \Big\{\sum_{i=1}^n a_i(x,\delta u)D_iT_k(u-w)\Big\}dx \leqslant \int\limits_\Omega F(x,u)\,T_k(u-w)\,dx. \tag{80}$$

Put

$$H=\{|u-w|<k\},\quad H_0=\{|u-w|=k\},$$

and let for every $l\in\mathbb{N}$

$$H_l=\{|u_l-w|<k\}\backslash H_0,\quad E_l=\{|u_l-w|<k\}\cap H_0.$$

First of all we prove that for every function $\varphi\in L^1(\Omega)$

$$\int\limits_{H_l}\varphi\,dx\to\int\limits_H\varphi\,dx. \tag{81}$$

Indeed, let $\varphi\in L^1(\Omega)$. For every $j\in\mathbb{N}$ put

$$H^{(j)}=\{|u-w|<k-1/j\},\qquad \tilde H^{(j)}=\{|u-w|>k+1/j\}.$$

We have

$$\operatorname{meas}(H\backslash H^{(j)})\to 0,\qquad \operatorname{meas}(\{|u-w|>k\}\backslash\tilde H^{(j)})\to 0. \tag{82}$$

We fix an arbitrary $\varepsilon>0$. In view of the property of Lebesgue integral's absolute continuity and (82) there exists $j\in\mathbb{N}$ such that

$$\int\limits_{H\backslash H^{(j)}}|\varphi|\,dx\leqslant\varepsilon/4,\qquad \int\limits_{\{|u-w|>k\}\backslash\tilde H^{(j)}}|\varphi|\,dx\leqslant\varepsilon/4. \tag{83}$$

Moreover, in view of the property of Lebesgue integral's absolute continuity, (37), and Egorov's theorem there exists a measurable set $\Omega'\subset\Omega$ such that

$$\int\limits_{\Omega\backslash\Omega'}|\varphi|\,dx\leqslant\varepsilon/4, \tag{84}$$

$$u_l\to u\ \text{ uniformly in }\Omega'. \tag{85}$$

Assertion (85) means that we can find $l_0\in\mathbb{N}$ such that for every $l\in\mathbb{N}$, $l\geqslant l_0$, and $x\in\Omega'$

$$|u_l(x)-u(x)|<1/j. \tag{86}$$

Let $l\in\mathbb{N}$, $l\geqslant l_0$. From (86) it follows that

$$(H^{(j)}\backslash H_l)\cap\Omega'=\emptyset,\qquad \{|u_l-w|<k\}\cap\tilde H^{(j)}\cap\Omega'=\emptyset.$$

Then

$$H\backslash H_l\subset(H\backslash H^{(j)})\cup(\Omega\backslash\Omega'),\qquad H_l\backslash H\subset(\{|u-w|>k\}\backslash\tilde H^{(j)})\cup(\Omega\backslash\Omega').$$

These facts, (83), and (84) imply that

$$\int\limits_{H\backslash H_l}|\varphi|\,dx\leqslant\varepsilon/2,\qquad \int\limits_{H_l\backslash H}|\varphi|\,dx\leqslant\varepsilon/2.$$

Hence,

$$\left|\int\limits_{H_l}\varphi\,dx-\int\limits_H\varphi\,dx\right|\leqslant\varepsilon.$$

The latter estimate means that (81) is true.

Further, put

$$k_1 = k + \|w\|_{L^\infty(\Omega)}, \quad \varphi_1 = \sum_{i=1}^{n} a_i(x, \nabla T_{k_1+1}(u))\, D_i w,$$

and let for every $l \in \mathbb{N}$

$$\psi_l = \sum_{i=1}^{n} a_i(x, \nabla u_l)\, D_i u_l + g_2,$$

$$S_l' = \int\limits_{H_l} \left\{ \sum_{i=1}^{n} \left[a_i(x, \nabla u_l) - a_i(x, \nabla T_{k_1+1}(u)) \right] D_i w \right\} dx,$$

$$S_l'' = \int\limits_{E_l} \left\{ \sum_{i=1}^{n} a_i(x, \nabla w)\, [D_i u_l - D_i w] \right\} dx.$$

We fix an arbitrary $l \in \mathbb{N}$. In view of (14) we have

$$\int\limits_{\Omega} \left\{ \sum_{i=1}^{n} a_i(x, \nabla u_l) D_i T_k(u_l - w) \right\} dx = \int\limits_{\Omega} (f_l - F_l(x, u_l))\, T_k(u_l - w) dx. \tag{87}$$

Using (2) and (6), we get

$$\int\limits_{\Omega} \left\{ \sum_{i=1}^{n} a_i(x, \nabla u_l) D_i T_k(u_l - w) \right\} dx \geq \int\limits_{H_l} \left\{ \sum_{i=1}^{n} a_i(x, \nabla u_l)\, [D_i u_l - D_i w] \right\} dx + S_l''.$$

From this inequality and (87) we obtain

$$\int\limits_{H_l} \left\{ \sum_{i=1}^{n} a_i(x, \nabla u_l)\, D_i u_l \right\} dx \leqslant \int\limits_{H_l} \left\{ \sum_{i=1}^{n} a_i(x, \nabla u_l)\, D_i w \right\} dx + \int\limits_{\Omega} (f_l - F_l(x, u_l))\, T_k(u_l - w)\, dx - S_l''.$$

Hence, for every $l \in \mathbb{N}$

$$\int\limits_{H_l} \psi_l\, dx \leqslant \int\limits_{\Omega} (f_l - F_l(x, u_l))\, T_k(u_l - w)\, dx + \int\limits_{H_l} (\varphi_1 + g_2)\, dx + S_l' - S_l''. \tag{88}$$

Note that by virtue of (12) and (37) we get $f_l\, T_k(u_l - w) \to f\, T_k(u - w)$ strongly in $L^1(\Omega)$. Therefore,

$$\int\limits_{\Omega} f_l\, T_k(u_l - w)\, dx \to \int\limits_{\Omega} f\, T_k(u - w)\, dx. \tag{89}$$

Besides, in view of (57) and (37) we obtain $F_l(x, u_l)\, T_k(u_l - w) \to (f - F(x, u))\, T_k(u - w)$ strongly in $L^1(\Omega)$. Hence,

$$\int\limits_{\Omega} F_l(x, u_l)\, T_k(u_l - w)\, dx \to \int\limits_{\Omega} (f - F(x, u))\, T_k(u - w)\, dx. \tag{90}$$

As $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu, \Omega)$, then we have $T_k(u) \in \overset{\circ}{W}{}^{1,q}(\nu, \Omega)$. Therefore, assertion a) of Proposition 3.1 implies an inclusion $\varphi_1 \in L^1(\Omega)$. Besides, we have $g_2 \in L^1(\Omega)$. Thus, using (81), we deduce that

$$\int\limits_{H_l} (\varphi_1 + g_2)\, dx \to \int\limits_{H} (\varphi_1 + g_2)\, dx. \tag{91}$$

Now we prove that

$$S_l' \to 0. \tag{92}$$

Indeed, let $\varepsilon \in (0,1)$. In view of the property of Lebesgue integral's absolute continuity, (37), (55), and Egorov's theorem there exists a measurable set $\Omega_1 \subset \Omega$ such that

$$\int\limits_{\Omega\setminus\Omega_1} \left\{|\varphi_1| + \sum_{i=1}^{n} \nu_i\, |D_i w|^{q_i}\right\} dx \leqslant \varepsilon^n, \tag{93}$$

$$u_l \to u \ \text{ uniformly in } \ \Omega_1, \tag{94}$$

$$\sum_{i=1}^{n} a_i(x, \nabla u_l)\, D_i w \to \sum_{i=1}^{n} a_i(x, \delta u)\, D_i w \quad \text{iniformly in } \ \Omega_1. \tag{95}$$

Assertion (94) means that we can find $l_0 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant l_0$, and $x \in \Omega_1$

$$|u_l(x) - u(x)| \leqslant \varepsilon. \tag{96}$$

Moreover, in view of (95) there exists $l_1 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant l_1$, we get

$$\int\limits_{\Omega_1} \left| \sum_{i=1}^{n} a_i(x, \nabla u_l)\, D_i w - \sum_{i=1}^{n} a_i(x, \delta u)\, D_i w \right| dx \leqslant \varepsilon. \tag{97}$$

Let $l \in \mathbb{N}$, $l \geqslant \max(l_0, l_1)$. As $w \in L^\infty(\Omega)$, there exists a set $\hat{E} \subset \Omega$, meas $\hat{E} = 0$, such that for every $x \in \Omega\setminus\hat{E}$ we have $|w(x)| \leqslant \|w\|_{L^\infty(\Omega)}$. From this fact and (96) it follows that $(H_l \cap \Omega_1)\setminus\hat{E} \subset \{|u| < k_1 + 1\}$. Using this inclusion, Proposition 2.4, and (97), we obtain

$$\int\limits_{H_l\cap\Omega_1} \left| \sum_{i=1}^{n} a_i(x, \nabla u_l)\, D_i w - \varphi_1 \right| dx \leqslant \varepsilon.$$

The latter inequality and (93) imply that

$$|S_l'| \leqslant 2\varepsilon + \sum_{i=1}^{n} \int\limits_{H_l\setminus\Omega_1} |a_i(x, \nabla u_l)|\, |D_i w|\, dx. \tag{98}$$

Taking into account Hölder inequality, (4), an inclusion $H_l\setminus\hat{E} \subset \{|u_l| < k_1\}$, (15), and (93), we established that for every $i \in \{1, \dots, n\}$

$$\int\limits_{H_l\setminus\Omega_1} |a_i(x, \nabla u_l)|\, |D_i w|\, dx \leqslant (c_1 c_3 k_1 + 1 + \|g_1\|_{L^1(\Omega)})\varepsilon.$$

From this and (98) we deduce

$$|S_l'| \leqslant 2\varepsilon + n(c_1 c_3 k_1 + 1 + \|g_1\|_{L^1(\Omega)})\varepsilon.$$

Thus, (92) is true.

Further, we show that

$$S_l'' \to 0. \tag{99}$$

It suffices to take meas $H_0 > 0$. Let $i \in \{1, ..., n\}$. Since $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu, \Omega)$ and $w \in \overset{\circ}{W}{}^{1,q}(\nu, \Omega) \cap L^\infty(\Omega)$, by virtue of Proposition 2.5, we have $u - w \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu, \Omega)$. Hence, from Proposition 2.4 it follows that

$$D_i T_k(u - w) = 0 \quad \text{a.e. in } \ H_0. \tag{100}$$

On the other hand, for almost every $x \in H_0$ the inequality $|u(x)| < k_1 + 1$ holds. So, $T_k(u - w) = T_{k_1+1}(u) - w$ a.e. in H_0. Therefore,

$$D_i T_k(u - w) = D_i T_{k_1+1}(u) - D_i w \quad \text{a.e. in } \ H_0.$$

Then, taking into account (100), we get $D_i T_{k_1+1}(u) = D_i w$ a.e. in H_0. This and Proposition 2.4 imply that $\delta_i u = D_i w$ a.e. in H_0. From this result and (41) we infer that for every $i \in \{1, \dots, n\}$ $D_i u_l \to D_i w$ a.e. in H_0. Hence,

$$\sum_{i=1}^{n} a_i(x, \nabla w)\,[D_i u_l - D_i w] \to 0 \quad \text{a.e. in } H_0. \tag{101}$$

Next, we put

$$\varphi_2 = \sum_{i=1}^{n} |a_i(x, \nabla w)|\,|D_i w|, \qquad \varphi_3 = \sum_{i=1}^{n} (1/\nu_i)^{1/(q_i-1)} |a_i(x, \nabla w)|^{q_i/(q_i-1)}.$$

In view of (4) the functions φ_2 and φ_3 are summable in Ω.

We fix an arbitrary $\varepsilon > 0$. In view of the property of Lebesgue integral's absolute continuity, (101), and Egorov's theorem there exists a measurable set $\Omega_2 \subset H_0$ such that

$$\int\limits_{H_0 \setminus \Omega_2} (\varphi_2 + \varphi_3)\,dx \leqslant \varepsilon, \tag{102}$$

$$\sum_{i=1}^{n} a_i(x, \nabla w)\,[D_i u_l - D_i w] \to 0 \quad \text{iniformly in } \Omega_2.$$

The latter property means that me can find $l_0 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant l_0$,

$$\int\limits_{\Omega_2} \left| \sum_{i=1}^{n} a_i(x, \nabla w)\,[D_i u_l - D_i w] \right| dx \leqslant \varepsilon. \tag{103}$$

Let $l \in \mathbb{N}$, $l \geqslant l_0$. Using (102) and (103), we infer that

$$|S_l''| \leqslant 2\varepsilon + \sum_{i=1}^{n} \int\limits_{E_l \setminus \Omega_2} |a_i(x, \nabla w)|\,|D_i u_l|\,dx. \tag{104}$$

By the virtue of Hölder inequality, (102), and (15) we deduce that for every $i \in \{1, \dots, n\}$

$$\int\limits_{E_l \setminus \Omega_2} |a_i(x, \nabla w)|\,|D_i u_l|\,dx \leqslant \left(\int\limits_{E_l \setminus \Omega_2} \varphi_3\,dx \right)^{(q_i-1)/q_i} \left(\int\limits_{\{|u_l|<k_1\}} \nu_i |D_i u_l|^{q_i}\,dx \right)^{1/q_i} \leqslant \varepsilon^{(q_i-1)/q_i} (c_3 k_1)^{1/q_i}.$$

This fact along with (104) and an arbitrariness of ε implies that (99) is true.

Further, let $\chi : \Omega \to \mathbb{R}$ be a characteristic function of the set H, and let for every $l \in \mathbb{N}$ $\chi_l : \Omega \to \mathbb{R}$ be a characteristic function of the set H_l. We have

$$\varliminf_{l \to \infty} \chi_l \geqslant \chi \quad \text{a.e. in } \Omega. \tag{105}$$

Indeed, in view of (37) there exists a set $E_0 \subset \Omega$, meas $E_0 = 0$, such that for every $x \in \Omega \setminus E_0$ $u_l(x) \to u(x)$. Let $x \in \Omega \setminus E_0$. If $x \notin H$, then $\chi(x) = 0$. Hence, $\chi(x) \leqslant \chi_l(x)$, $\forall\, l \in \mathbb{N}$. Let $x \in H$. As $u_l(x) \to u(x)$, there exists $l_1 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant l_1$, we have $|u_l(x) - u(x)| < k - |u(x) - w(x)|$. Then for arbitrary $l \in \mathbb{N}$, $l \geqslant l_1$, we get $|u_l(x) - w(x)| < k$. Therefore, $x \in H_l$ and $\chi_l(x) = 1 = \chi(x)$. Thus, in any case we have $\chi(x) \leqslant \varliminf\limits_{l \to \infty} \chi_l(x)$ and assertion (105) holds.

From (105), (41), (55), and (5) it follows that

$$\varliminf_{l \to \infty} (\psi_l \chi_l) \geqslant \left(\sum_{i=1}^{n} a_i(x, \delta u) \delta_i u + g_2 \right) \chi \quad \text{a.e. in } \Omega. \tag{106}$$

Using (5), (88)–(92), (99), Fatou's lemma, and (106), we established that the function $\left(\sum_{i=1}^{n} a_i(x,\delta u)\delta_i u + g_2\right)\chi$ is summable in Ω and

$$\int_\Omega \left\{\sum_{i=1}^{n} a_i(x,\delta u)\delta_i u + g_2\right\}\chi\, dx \leqslant \int_\Omega F(x,u)\, T_k(u-w)\, dx + \int_H (\varphi_1 + g_2)\, dx.$$

From the latter inequality and Propositions 2.4 and 2.5 we obtain (80).

So, we proved that $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(v,\Omega)$, and properties (9) and (10) of Definition 2.2 are satisfied. Thus, u is an entropy solution to the Dirichlet problem (7), (8). The theorem is proved. □

References

[1] Bénilan Ph. et al., An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 1995, 22, No. 2, 241–273

[2] Alvino A., Ferone V., Trombetti G., Nonlinear elliptic equations with lower-order terms, Differential Integral Equations, 2001, 14, No. 10, 1169–1180

[3] Betta M.F., Mercaldo A., Murat F., Porzio M.M., Existence of renormalized solutions to nonlinear elliptic equations with a lower-order term and right-hand side a measure, J. Math. Pures Appl. (9), 2003, 82, No. 1, 90–124

[4] Boccardo L., Gallouët T., Nonlinear elliptic and parabolic equations involving measure data, J. Funct. Anal., 1989, 87, No. 1, 149–169

[5] Boccardo L., Gallouët T., Nonlinear elliptic equations with right hand side measures, Comm. Partial Differential Equations, 1992, 17, No. 3-4, 641–655

[6] Boccardo L., Gallouët T., Summability of the solutions of nonlinear elliptic equations with right hand side measures, J. Convex Anal., 1996, 3, No. 2, 361–365

[7] Boccardo L., Gallouët T., Orsina L., Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1996, 13, No. 5, 539–551

[8] Dal Maso G., Murat F., Orsina L., Prignet A., Renormalized solutions of elliptic equations with general measure data, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 1999, 28, No. 4, 741–808

[9] Kovalevsky A.A., On a sharp condition of limit summability of solutions of nonlinear elliptic equations with L^1-right-hand sides, Ukr. Math. Bull., 2005, 2, No. 4, 507–545

[10] Kovalevsky A.A., General conditions for limit summability of solutions of nonlinear elliptic equations with L^1-data, Nonlinear Anal., 2006, 64, No. 8, 1885–1895

[11] Boccardo L., Gallouët T., Marcellini P., Anisotropic equations in L^1, Differential Integral Equations, 1996, 9, No. 1, 209–212

[12] Bendahmane M., Karlsen K.H., Nonlinear anisotropic elliptic and parabolic equations in R^N with advection and lower order terms and locally integrable data, Potential Anal., 2005, 22, No. 3, 207–227

[13] Bendahmane M., Karlsen K.H., Anisotropic nonlinear elliptic systems with measure data and anisotropic harmonic maps into spheres, Electron. J. Differential Equations, 2006, No. 46, 30 pp.

[14] Li F.Q., Anisotropic elliptic equations in L^m, J. Convex Anal., 2001, 8, No. 2, 417–422

[15] Aharouch L., Azroul E., Benkirane A., Quasilinear degenerated equations with L^1 datum and without coercivity in perturbation terms, Electron. J. Qual. Theory Differ. Equ., 2006, 19, 18 pp.

[16] Atik Y., Rakotoson J.-M., Local T-sets and degenerate variational problems, Appl. Math. Lett., 1994, 7, No. 4, 49–53

[17] Cavalheiro A.C., Existence of entropy solutions for degenerate quasilinear elliptic equations, Complex Var. Elliptic Equ., 2008, 53, No. 10, 945–956

[18] Cirmi G.R., On the existence of solutions to non-linear degenerate elliptic equations with measures data, Ricerche Mat., 1993, 42, No. 2, 315–329

[19] Li F.Q., Nonlinear degenerate elliptic equations with measure data, Comment. Math. Univ. Carolin., 2007, 48, No. 4, 647–658

[20] Kovalevsky A.A., Gorban Yu.S., Solvability of degenerate anisotropic elliptic second-order equations with L^1-data, Electron. J. Differential Equations, 2013, No. 167, 1–17

[21] Kovalevsky A.A., Gorban Yu.S., Degenerate anisotropic variational inequalities with L^1-data, Institute of Applied Mathematics and Mechanics of NAS of Ukraine, 2007, Donetsk, preprint No. 2007.01, 92 pp. (in Russian)

[22] Kovalevsky A.A., Gorban Yu.S., Degenerate anisotropic variational inequalities with L^1-data, C. R. Math. Acad. Sci. Paris, 2007, 345, No. 8, 441–444

[23] Kovalevsky A.A., Gorban Yu.S., On T-solutions of degenerate anisotropic elliptic variational inequalities with L^1-data, Izv. Math., 2011, 75, No. 1, 101–160 (in Russian)

[24] Kovalevsky A.A., Entropy solutions of Dirichlet problem for a class of nonlinear elliptic fourth order equations with L^1-right-hand sides, Izv. Math., 2001, 65, No. 2, 27–80 (in Russian)

[25] Kinderlehrer D., Stampacchia G., An introduction to variational inequalities and their applications, 1980, United Kingdom Edition, Academic Press, New York-London
[26] Lions J.-L., Some methods for solving nonlinear boundary value problems, 1972, Mir, Moscow (in Russian)